

CONVEX AND CONCAVE DECOMPOSITIONS OF AFFINE 3-MANIFOLDS

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Dedicated to the 60th Birthday of Bill Goldman

ABSTRACT. An (flat) affine 3-manifold is a 3-manifold with an atlas of charts to an affine space \mathbb{R}^3 with transition maps in the affine transformation group $\mathbf{Aff}(\mathbb{R}^3)$. We will show that a closed affine 3-manifold is either an affine Hopf 3-manifold or decomposes canonically to concave affine submanifolds with incompressible boundary, toral π -submanifolds and 2-convex affine manifolds, each of which is an irreducible 3-manifold. This will help us with understanding the affine 3-manifolds better.

1. INTRODUCTION

1.1. Introduction and history. An [affine manifold](#) is a manifold with an atlas of charts to \mathbb{R}^n , $n \geq 2$, where the transition maps are in the affine group. Euclidean manifolds are examples. A [Hopf manifold](#) that is the quotient of $\mathbb{R}^n - \{O\}$ by a linear contraction group, i.e., a group of linear transformation generated by an element of with eigenvalues of norm greater than 1 is an example. A half-Hopf manifold is the quotient of $H - \{O\}$ by a linear contraction group for a closed upper half-space H of \mathbb{R}^n . (See Proposition [2.5](#).) Any real projective manifold projectively diffeomorphic to one of these is called by the same name.

Recall that the affine space \mathbb{R}^n is imbedded in $\mathbb{R}P^n$ as a complement of a hyperspace, and affine transformation groups naturally extend. The hyperspace is the set of ideal points $\mathbb{R}P_\infty^n$ of \mathbb{R}^n . Affine geodesics also extends to projective geodesics.

We will look an affine manifold as a [real projective manifold](#), i.e., a manifold with an atlas of charts to $\mathbb{R}P^n$ with transition maps in the projective group $\mathrm{PGL}(n + 1, \mathbb{R})$. An affine manifold has a canonical

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real projective structure since the charts and the transition maps are projective also.

Closed affine 2-manifolds were classified by Benzécri [11] and Nagano-Yagi [37]. Benzécri [10] made some initial stepping stones in this area. For the currently most extensive set of examples of affine manifolds, see the paper by Sullivan and Thurston [41]. We still have not obtained essentially different examples to theirs to this date. (See also Carrière [14], Smillie [40] and Benoist [7] and [9].) A compact affine 3-manifold is *radiant* if the holonomy group fixes a unique point. (See Section 2.2 and Barbot [5], Fried, Goldman, and Hirsch [27].) Such a manifold has a complete flow called a *radiant flow*. A generalized affine suspension is a radiant affine manifold admitting a total cross-section. (See Proposition 2.2.) A radiant affine n -manifold can be constructed easily from a real projective $(n - 1)$ -manifold using generalized affine suspension. (See Section 2.2 of [5] or Chapter 3 of [17].)

An affine manifold affinely diffeomorphic to \mathbb{R}^n/Γ for a discrete subgroup Γ of $\mathbf{Aff}(\mathbb{R}^n)$ is called a *complete affine manifold*. Recall that Markus conjectured in the 1970s that a closed affine manifold with a parallel volume form is complete. The conjecture is still open for $n \geq 3$. Also, there is a well-known conjecture that a closed hyperbolic 3-manifold does not admit an affine structure. Fried [25] showed that hyperbolic Dehn-surgered 3-manifolds from a figure-eight knot in \mathbb{S}^3 do not admit affine structures. When the holonomy group is solvable or nilpotent, we know a great deal by the work of Fried, Goldman and Hirsch [27], Benoist [9] and Dupont [24]. Otherwise, very little is known for affine 3-manifolds except for the following: The classification of complete affine 3-manifolds by the work of Fried and Goldman [26] and that of the closed radiant 3-dimensional ones by Barbot [5] and Choi [17] answering the question of Carrière and Fried. We showed that a closed radiant affine 3-manifolds have total cross-sections to the radial flow (See Theorem 2.3.) We refer to [6] and [17].

Remark 1.1. *We mention an error in [17] for Theorems A and Corollary A and B. We assume that not only that the holonomy group of the affine manifold M fixes a common point but we need that the boundary is tangent to the radial vector field. Proposition 2.1 should fill in the gap since we just need to use the fact that radial lines foliate the universal cover. For closed radiant affine manifolds, everything is fine in [17].*

Also, in 1960s the Auslander conjecture that complete affine n -manifolds have virtually solvable fundamental groups is solved for $n = 3$ by Fried and Goldman [26], and for $n \leq 6$ by Abels, Margulis, and

Soifer [1], [2], [3]. The Chern conjecture that a closed affine manifold has Euler characteristic 0 is solved only for complete closed affine manifolds by the work of Kostant and Sullivan [36] and for closed affine manifolds with amenable holonomy groups by Hirsch and Thurston [32].

However, the complete affine manifolds are studied often with very different techniques using Lie groups. The study of incomplete affine manifolds is very geometric.

Our main subject is the following: The question of Goldman in Problem 6 in the Open problems section of [4] is whether closed affine 3-manifolds are prime. We showed that 2-convex affine 3-manifolds are irreducible in [18]. Wu in his doctoral thesis researched into this topic [46] in 2012 and showed that the sphere used in the connected sum cannot be affinely diffeomorphic to an imbedded sphere in \mathbb{R}^3 . Our Theorem 1.4 shows that closed affine manifolds may be obtainable by gluing a solid torus or a solid Klein bottle with special geometric properties to an irreducible 3-manifold. This construction may result in reducible 3-manifolds as we can see from Gordon [29]. Hence, the existence of solid torus or solid Klein bottles with special geometric properties gives us difficulty in proving this conjecture.

For the related real projective structures on closed 3-manifolds, Cooper and Goldman [23] showed that a connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$ admits no real projective structure. For a recent survey on real projective structures centered on convex ones and deformations, see Choi-Lee-Marquis [22]. For these topics, a good reference is given by Goldman [28], originally given as lecture notes in 1980s.

1.2. Main results. We use the double-covering map $\mathbb{S}^3 \rightarrow \mathbb{R}P^3$, and hence \mathbb{S}^3 has a real projective structure. The group of projective automorphism of $\mathbb{R}P^3$ is $\mathrm{PGL}(4, \mathbb{R})$ and that of \mathbb{S}^3 is $\mathrm{SL}_{\pm}(4, \mathbb{R})$. We will model our manifold on \mathbb{S}^3 and the group of projective automorphisms on it.

We recall the main results of [19] which we will state in Section 2.4 in a more detailed way. Let M be a closed real projective manifold. Let \tilde{M} be the universal cover and $\pi_1(M)$ the deck transformation group. A projective structure on M gives us an immersion $\mathbf{dev} : \tilde{M} \rightarrow \mathbb{R}P^3$ equivariant with respect to a homomorphism $h : \pi_1(M) \rightarrow \mathrm{PGL}(4, \mathbb{R})$. This data is given by the real projective structure.

Let J be a subgroup of the kernel of h , normal in $\pi_1(M)$. We cover M by a regular cover $M_J = \tilde{M}/J$ corresponding to J with

- an induced and lifted immersion $\mathbf{dev}_J : M_J \rightarrow \mathbb{S}^3$ and

- induced holonomy homomorphism $h_J : \pi_1(M)/J \rightarrow \mathrm{SL}_\pm(4, \mathbb{R})$ satisfying

$$\mathbf{dev}_J \circ g = h_J(g) \circ \mathbf{dev}_J \text{ for } g \in \pi_1(M)/J.$$

We obtain a Cauchy completion \check{M}_J of M_J by pulling back the path metric of the Fubini-Study Riemannian metric of \mathbb{S}^3 , called a *Kuiper completion*. The *ideal set* is $M_{J,\infty} := \check{M}_J - M_J$.

A 3-hemisphere is a closed 3-hemisphere in \mathbb{S}^3 and a 3-bihedron is the closure of a component $H - \mathbb{S}^2$ for a 3-hemisphere H with a great 2-sphere \mathbb{S}^2 passing H° . These have real projective structures induced from the double-covering map $\mathbb{S}^3 \rightarrow \mathbb{RP}^3$.

If the universal cover \tilde{M} is projectively diffeomorphic to an open hemisphere, i.e., \mathbb{R}^n , then M is called a *complete affine manifold*. If the universal cover \tilde{M} is projectively diffeomorphic to an open 3-bihedron, we call M a *bihedral real projective manifold*. Again the subject of classifying such real projective manifold is completely open.

We will explain these later in more detail in Section 2.4. A *hemispherical crescent* is a 3-hemisphere in \check{M}_J with boundary 2-hemisphere in the ideal set. A *bihedral 3-crescent* is a 3-bihedron B in \check{M}_J so that a boundary 2-hemisphere is the ideal set where we assume that \check{M}_J has no hemispherical 3-crescent. (See Hypothesis 2.10.) A *concave affine 3-manifold* is a codimension-zero compact submanifold of M defined in [19]. We cover these in Section 2.4.3.

- A concave affine 3-manifold of type I is an affine manifold covered by $R \cap M_J$ for a hemispherical crescent R (Definition 2.12).
- A concave affine 3-manifold of type II is an affine manifold covered by $U \cap M_J$ of a union U of bihedral crescents in M_J extending their open ideal boundary 2-hemispheres. (Definition 2.13.)

Both are called concave affine manifolds. The interior of a concave affine 3-manifold has an affine structure inducing its real projective structure. A *two-faced submanifold of type I* of a real projective 3-manifold M is roughly given as the totally geodesic submanifold arising from the intersection in M_J of two hemispherical crescents meeting only in the boundary. A *two-faced submanifold of type II* of a real projective 3-manifold M is roughly given as the totally geodesic submanifold arising from the intersection in M_J of two bihedral crescents meeting only in the boundary. Both are called *two-faced submanifold*. (Note here we use the term “two-faced” in a different way from commonly used “two-sided” as in manifold topology.)

Let T be a convex simplex in an affine space \mathbb{R}^3 with faces F_0, F_1, F_2 , and F_3 . A real projective or affine 3-manifold is *2-convex* if every projective map $f : T^o \cup F_1 \cup F_2 \cup F_3 \rightarrow M$ extends to $f : T \rightarrow M$. (Y. Carrière [14] first defined this concept.)

Theorem 1.2 ([19]). *Suppose that M is a compact real projective 3-manifold with empty or convex boundary. Suppose that M is not 2-convex. Then \check{M}_J contains a hemispherical or bihedral 3-crescent.*

Now, we sketch the process of canonical decomposition in [19]:

- Suppose that a hemispherical crescent $R \subset \check{M}_J$ exists.
 - If there is a two-faced submanifold of type I, then we can [split](#) M along this submanifold to obtain M^s . If not, we let $M^s = M$. Let M_J^s denote the corresponding cover of M^s obtained by splitting M_J and taking a component and \check{M}_J^s its Kuiper completion.
 - Then hemispherical crescents in \check{M}_J^s are mutually disjoint and cover mutually disjoint collection of compact submanifolds, called [concave affine manifolds of type I](#).
 - We remove all these. Then we let the resulting compact manifold be called $M^{(1)}$. The boundary is still convex.
- Let $M_J^{(1)}$ denote the cover of $M^{(1)}$ obtained by removing corresponding submanifolds from M_J^s , and let $\check{M}_J^{(1)}$ the Kuiper completion. Suppose that there is a bihedral crescent $R \subset \check{M}_J^{(1)}$.
 - If there is a two-faced submanifold of type II, then we can [split](#) $M^{(1)}$ along this submanifold to obtain $M^{(1)s}$. If not, we let $M^{(1)s} = M^{(1)}$.
 - Let $M_J^{(1)s}$ denote the cover of $M^{(1)s}$ obtained by $M_J^{(1)}$ by splitting and taking a component, and let $\check{M}_J^{(1)s}$ the Kuiper completion. Then the union of bihedral crescents in $\check{M}_J^{(1)s}$ covers the union of mutually disjoint collection of compact submanifolds, called [concave affine manifolds of type II](#).
 - We remove all these. Then the resulting compact real projective manifold $M^{(2)}$ with convex boundary is 2-convex.

We will further sharpen the result in this paper. A [toral \$\pi\$ -submanifold](#) is a compact radiant concave affine 3-manifold with the virtually infinite-cyclic fundamental group covered by a special domain in a hemisphere. A half-Hopf 3-manifold is an example, probably the most typical one. We will later show that a toral π -submanifold is homeomorphic to a solid torus or a solid Klein bottle. (See Definition 3.6 and Lemmas

3.12 and 3.13.) In fact, we do not understand how half-affine Hopf 3-manifolds may exist in an isolated manner in a closed affine 3-manifold, which is the main unresolved difficult question.

Theorem 1.3. *Let M be a connected compact real projective 3-manifold with empty or convex boundary.*

- *Let M^s be the resulting real projective 3-manifold after [splitting](#) along two-faced totally geodesic submanifolds of type I (resp. of type II).*
- *Let N be a compact concave affine 3-manifold in M^s with compressible boundary of type I (resp. of type II).*

Then N contains a unique toral π -submanifold of type I (resp. of type II) or M is an [affine Hopf 3-manifold](#).

The theorem works with all type I objects or all type II objects in the above statements.

By Theorem 3.15, concave affine 3-manifolds are irreducible. Theorem 0.1 of [18] shows that a 2-convex affine 3-manifold is covered by a cell and hence is irreducible.

So far, our results are on real projective 3-manifolds. Now we go over to the result specific to affine 3-manifolds. For affine 3-manifolds, \tilde{M}_J contains no hemispherical crescents since the developing map goes into the affine space \mathbb{H} strictly.

Theorem 1.4. *Let M be a connected compact affine 3-manifold with empty or convex boundary. Let M^s denote the M split along two-faced submanifolds of type II. Then either M is an [affine Hopf 3-manifold](#) or M^s decomposes into compact submanifolds as follows:*

- *a 2-convex affine 3-manifold with convex boundary,*
- *toral π -submanifolds of type II in a concave affine 3-manifold with compressible boundary with the virtually cyclic holonomy group, or*
- *concave affine 3-manifolds of type II with boundary incompressible to themselves.*

All of these submanifolds obtained above are prime 3-manifolds.

However, the above decomposition is not necessarily a prime decomposition.

We do not yet have any examples of nonconvex 2-convex affine 3-manifolds except for radiant ones nor a good idea how to classify 2-convex affine 3-manifolds with convex boundary. Some examples in Sullivan-Thurston [41] as the affine 4-manifolds fibering over surfaces

are 2-convex. By projectivizing, we obtain 2-convex real projective 3-manifolds. (See also [19] and [31].)

Also, we are trying to understand the concave affine 3-manifolds with boundary incompressible into themselves. Since the boundary is concave for a concave affine 3-manifold, we don't have a clear picture yet. Actually, we wish to obtain a more precise canonical decomposition theorem of an affine 3-manifold into concave affine 3-manifolds and 2-convex affine 3-manifolds similar to what we did for real projective surfaces as in [20], [21].

Also, we are attempting to classify the toral π -submanifolds which might be more tractable. Answers to these questions would solve Goldman's question.

1.3. Outline. We outline this paper. The main tools of this paper are from three long papers [19], [17], and [18]. In Section 2, we discuss the basic facts on real projective and affine structures. In Section 2.3, we prove various facts about affine Hopf manifolds and half-Hopf manifolds. In Section 2.4, we recall the convex and concave decomposition of real projective structures. We recall 3-crescents and two-faced submanifolds and the decomposition theory in [19].

In Section 3, Theorem 3.2 claims that if we have a two-faced submanifold that is compressible, then the manifold is finitely covered by an affine Hopf 3-manifold. The idea for the proof is by a so-called disk-fixed-point argument, Proposition A.3; that is, we can find an attracting fixed point of a deck transformation g using a simple closed curve c bounding a disk D with $g(c) \subset D^\circ$. We prove Theorem 3.2 in Section 3.2.

We prove Theorems 3.7 and 3.8 in Section 3.3 using Lemma 3.10. Here, we show that a cover of the concave affine 3-manifold being a union of mutually intersecting 3-crescents must map to a domain in a hemisphere by dev_J , and the boundary has a unique annulus component. Since the fundamental group of N acts on an annulus covering its boundary properly and freely, the fundamental group is virtually infinite-cyclic. The final part of the proof is completed by Section 3.4 where we show that these concave affine 3-manifolds contain toral π -submanifolds. We also show that a toral π -submanifold is homeomorphic to a solid torus or a solid Klein bottle.

In Section 4, we discuss the decomposition of M into 2-convex real projective 3-manifolds with convex boundary and toral π -submanifolds, i.e., Theorem 4.1. We use the convex and concave decomposition theorem of [19] and Theorems 3.7 and 3.8 and replacing the compact

concave affine 3-manifolds with compressible boundary with toral π -suborbifolds. We prove Theorem 1.3 lastly here.

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2. PRELIMINARY

2.1. The projective geometry of the sphere. Let V be a vector space. Define $P(V)$ as $V - \{0\} / \sim$ where $x \sim y$ iff $x = sy$ for $s \in \mathbb{R} - \{0\}$. $\mathrm{PGL}(V)$ acts on this space where $\mathrm{PGL}(\mathbb{R}^n) = \mathrm{PGL}(n, \mathbb{R})$.

Recall that $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$. A subspace of $\mathbb{R}P^n$ is the image $V - \{O\}$ of a subspace V of \mathbb{R}^{n+1} under the projection. The group of projective automorphisms is $\mathrm{PGL}(n+1, \mathbb{R})$ acting on $\mathbb{R}P^{n+1}$ in the standard manner. A *real projective n -manifold with empty or convex boundary* is given by a manifold with empty or nonempty boundary and an atlas of charts to $\mathbb{R}P^n$ and transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$ so that either the boundary is empty or each point of the boundary has a chart to a domain with boundary in $\mathbb{R}P^n$. The maximal atlas is called a *real projective structure*. The boundary is *totally geodesic* if each boundary point has a neighborhood projectively diffeomorphic to an open set in a half-space of an affine space meeting the boundary. A map between two real projective 3-manifolds are *projective* if the map is projective under the charts in the real projective structures. Such a structure gives us a pair (\mathbf{dev}, h) where $\mathbf{dev} : \tilde{M} \rightarrow \mathbb{R}P^n$ is an immersion equivariant with respect to $h : \pi_1(M) \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$. The image of h is called the *holonomy group*. Conversely, such a pair will determine a real projective structure. (See [19] for more detail.)

A subspace of $\mathbb{R}P^n$ is the image $V - \{O\}$ of a subspace V of \mathbb{R}^{n+1} under the projection. An affine space is \mathbb{R}^n with a group $\mathbf{Aff}(\mathbb{R}^n)$ of affine transformations of form $x \mapsto Mx + b$ for $M \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The complement A of $\mathbb{R}P^n - V$ for a subspace V of codimension-one can be identified with the affine space \mathbb{R}^n and the group $\mathbf{Aut}(A)$ of the projective transformations of A equals $\mathbf{Aff}(A)$. (Berger [12] gives many geometrical ideas around this.)

An *affine n -manifold with empty or convex boundary* is an n -manifold with smooth boundary and an atlas of charts to open subsets or convex domains in \mathbb{R}^n and the transition maps in $\mathbf{Aff}(\mathbb{R}^n)$. Since the affine transformations are projective, an affine n -manifold has a canonical real projective structure. Such n -manifolds are considered as real projective n -manifolds with special structures in this paper. A real projective manifold projectively homeomorphic to an affine manifold is called an *affine manifold* in this paper. (Note that there may be more than one

compatible affine structure. However, there is no such affine manifold in the paper.)

An elementary example is an *affine Hopf n -manifold* that is the quotient of $\mathbb{R}^n - \{O\}$ by an infinite-cyclic group generated by a linear map g all of whose eigenvalues have norm > 1 or by $\langle g, -I \rangle$ for g as above. The quotient is a manifold by Proposition A.2.

By Proposition 2.5, an affine Hopf 3-manifold is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, or a nonorientable \mathbb{S}^2 -bundle over \mathbb{S}^1 .

If g acts on an $(n-1)$ -plane passing O , and the half-space H in \mathbb{R}^n bounded by it, then $(H - \{O\})/\langle g \rangle$ is called a *half-Hopf 3-manifold*. A real projective manifold projectively homeomorphic to an affine Hopf n -manifold or half-Hopf n -manifold is called by the same name in this paper. (See [33] for a conformally flat version and [41].)

Let $\mathbb{R}_+ := \{t | t \in \mathbb{R}, t > 0\}$. Define $\mathbb{S}(V)$ as $V - \{0\} / \sim$ where $x \sim y$ iff $x = sy$ for $s \in \mathbb{R}_+$. $\mathbf{SL}_\pm(V)$ acts on $\mathbb{S}(V)$. There is a double cover $\mathbb{S}(V) \rightarrow P(V)$ with the deck transformation group generated by the antipodal map $\mathcal{A} : \mathbb{S}(V) \rightarrow \mathbb{S}(V)$ induced from the linear map $V \rightarrow V$ given by $v \rightarrow -v$. We denote by $\langle\langle v \rangle\rangle$ the equivalence class of v in $\mathbb{S}(V)$. This gives us *homogeneous coordinates* of $\mathbb{S}(\mathbb{R}^n)$ as $\langle\langle x_1, \dots, x_n \rangle\rangle$ for the vector $(x_1, \dots, x_n) \neq 0$. We denote by \mathbb{S}^3 the space $\mathbb{S}(\mathbb{R}^4)$.

The real projective sphere \mathbb{S}^3 has a real projective structure given by the double covering map to $\mathbb{R}P^3$. Let $\mathbf{Aut}(\mathbb{S}^3)$ denote the group of projective automorphisms of \mathbb{S}^3 . We can identify it with $\mathbf{SL}_\pm(4, \mathbb{R})$ as obtained by the standard action of $\mathbf{GL}(4, \mathbb{R})$ on \mathbb{R}^4 .

We imbed \mathbb{R}^3 as an open 3-hemisphere \mathbb{H}° in \mathbb{S}^3 for a closed 3-hemisphere \mathbb{H} by sending (x_1, x_2, x_3) to $\langle\langle 1, x_1, x_2, x_3 \rangle\rangle$. We identify \mathbb{R}^3 with \mathbb{H}° . The boundary of \mathbb{R}^3 is a great sphere \mathbb{S}_∞^2 given by $x_0 = 0$. The group $\mathbf{Aut}(\mathbb{H})$ of projective automorphisms acting on \mathbb{H} equals the group of affine transformations of $\mathbb{H}^\circ = \mathbb{R}^3$. (A good reference for all these geometric topics is the book by Berger [12].)

2.2. Radiant affine n -manifolds. Given any affine coordinates $x_i, i = 1, \dots, n$, of \mathbb{R}^n , a vector field $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is called a *radiant* vector field. O of the coordinate system is called the *origin* of the radiant vector field. Suppose that the holonomy group of an affine n -manifold M fixes an origin of an affine coordinate system. Then $\mathbf{dev}_J : M_J \rightarrow \mathbb{R}^n$ is an immersion and the radiant vector field lifts to a vector field in M_J . The vector field is invariant under the deck transformations of M_J and hence induces a vector field on M . The vector field on M is also called a *radiant vector field*. (see Barbot [5] and Chapter 3 of [17].) This gives us a *radiant* flow:

$$\mathbb{R} \times M \rightarrow M.$$

Let M be a radiant affine manifold with the holonomy group fixing a point O . A *radial line* in M_J is an arc α in M_J so that $\mathbf{dev}|_\alpha$ is an imbedding to a component of a complete real line l with O removed.

Proposition 2.1. *Let M be a compact affine n -manifold with empty or boundary. Suppose that the holonomy group fixes the origin of a radiant vector field and the boundary is tangent to the radiant vector field. Then $\mathbf{dev}_J(M_J)$ misses the origin of a vector field and M_J is foliated by radial lines.*

Proof. See the proof of Proposition 2.4 of Barbot [5]. \square

Let $\|\cdot\|$ denote the Euclidean metric of \mathbb{R}^n . Given a real projective $(n-1)$ -manifold Σ and a projective automorphism $\phi : \Sigma \rightarrow \Sigma$, we can obtain a radiant affine n -manifold homeomorphic to the mapping torus

$$\Sigma \times I / \sim \text{ where for every } x \in \Sigma, (x, 1) \sim (\phi(x), 0).$$

Let $\mathbf{dev} : \tilde{\Sigma} \rightarrow \mathbb{S}^{n-1}$ be a developing map with holonomy homomorphism $h : \pi_1(\Sigma) \rightarrow \mathbf{SL}_\pm(n, \mathbb{R})$. Then we extend \mathbf{dev} to

$$\mathbf{dev}' : \tilde{\Sigma} \times \mathbb{R} \rightarrow \mathbb{R}^n \text{ by } (x, t) \mapsto \exp(t)\mathbf{dev}(x).$$

The automorphism ϕ lifts to $\tilde{\phi} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ so that $\psi \circ \mathbf{dev} = \mathbf{dev} \circ \tilde{\phi}$ for $\psi \in \mathbf{SL}_\pm(n, \mathbb{R})$. For each element γ of $\pi_1(\Sigma)$, we define the action of $\pi_1(\Sigma)$ on $\Sigma \times \mathbb{R}$ by

$$\gamma(x, t) = (\gamma(x), \log \|h(\gamma)(x)\| + t).$$

This preserves the affine structure and the radial vector field. We also act by

$$\tilde{\phi} : \tilde{\Sigma} \times \mathbb{R} \rightarrow \tilde{\Sigma} \times \mathbb{R} \text{ by } \hat{\phi}(x, t) = (\tilde{\phi}(x), \log \|\psi(x)\| + t).$$

Then the result $\tilde{\Sigma} \times \mathbb{R} / \langle \hat{\phi}, \pi_1(\Sigma) \rangle$ is homeomorphic to the mapping torus. We call this construction or the manifold the *generalized affine suspension*.

Proposition 2.2. *A compact radiant affine n -manifold is a generalized affine suspension if and only if the radial flow has a total cross section.*

Theorem 2.3 (Barbot-Choi [17]). *Let M be a compact radiant affine 3-manifold with empty or totally geodesic boundary, and the boundary is tangent to the radiant vector field. Then M admits a total cross-section to the radiant flow.*

For $n = 6$, there is a counter-example due to D. Fried.

2.3. The infinite-cyclic holonomy group affine 3-manifolds. First, we will explore the affine Hopf manifolds.

Lemma 2.4. *Let X be an open manifold with a group G acting on it properly discontinuously and cocompactly. Let Σ be a compact connected submanifold where $X - \Sigma$ has two open components, and let U be a component. Then there exists an infinite-order element $g \in G$ so that $g(\Sigma) \subset U$ and $g(U) \subset U$.*

Proof. Let $x \in \Sigma$. Then there exists an infinite sequence $g_i(x)$ so that $g_i \in G$ is $g_i(x) \in U$. By proper-discontinuity, $g_i(\Sigma) \cap \Sigma = \emptyset$ except for finitely many i . We may choose g_i so that $g_i(U)$ is a proper subset of U . This property implies that g_i must be of infinite order. \square

Proposition 2.5. *An affine Hopf 3-manifold M is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, or a nonorientable \mathbb{S}^2 -bundle over \mathbb{S}^1 , only three up to the homeomorphism types. A half-Hopf 3-manifold M is homeomorphic to a solid torus or a solid Klein bottle.*

Proof. Let M be an affine Hopf 3-manifold. The universal cover is $\mathbb{R}^3 - \{O\}$ and hence M does not contain any fake cells. We double cover it so that it has an infinite cyclic holonomy group and call the double cover by M' . Let g be the generator of the holonomy group. Each eigenvalue of a nonidentity element $g \in h(\pi_1(M))$ has either all norms > 1 or all norm < 1 by definition. By taking g^{-1} if necessary, we assume that all the norms are < 1 . Let S be a unit sphere for a norm in Lemma A.1. By Lemma A.1, S and $g(S)$ are disjoint. Then S and $g(S)$ bound compact space homeomorphic to $S \times I$. We introduce an equivalence relation \sim where $x \sim y$ for $x \in S, y \in g(S)$ if $y = g(x)$. Thus,

$$(\mathbb{R}^3 - \{O\})/\langle g \rangle$$

is an \mathbb{S}^2 -bundle over \mathbb{S}^1 . Since $\text{Mod}(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$ is a classical work of Smale [39], there exists only two homeomorphism types of \mathbb{S}^2 -bundle over \mathbb{S}^1 .

Now, M is doubly or quadruply covered by $\mathbb{S}^2 \times \mathbb{S}^1$. Since $-I$ acts on S above, and $\text{Mod}(\mathbb{R}P^2) = 1$, the proposition is proved. \square

See Section 3 of [18] for the definition of the generalized affine suspension.

Theorem 2.6. *Let M be a compact affine 3-manifold with empty or totally geodesic boundary and a virtually infinite-cyclic holonomy group whose infinite-order generator fixes a point in the affine space. Then*

- M is finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ or $D^2 \times \mathbb{S}^1$.

- M is a generalized affine suspension of a sphere, $\mathbb{R}P^2$, or a 2-hemisphere.
- If M is closed, then M is an affine Hopf 3-manifold and is diffeomorphic to an \mathbb{S}^2 -bundle over \mathbb{S}^1 or $\mathbb{R}P^2 \times \mathbb{S}^1$. If M has totally geodesic boundary, then M is a half-Hopf manifold.
- Any 3-manifold covered by an affine Hopf 3-manifold or a half-Hopf 3-manifold is one also.

Proof. We take a finite cover N so that N has an infinite cyclic fundamental group. By Theorem 5.2 of [30], N has to be covered by $\mathbb{S}^2 \times \mathbb{S}^1$ or $D^2 \times \mathbb{S}^1$ finitely. Therefore, the universal cover \tilde{M} is not complete affine or bihedral.

By taking a finite cover N of M , we may assume that $h(\pi_1(N)) = \langle g \rangle$ and g fixes a point x in the affine space. Thus the holonomy group fixes a point x . Then N is a radiant affine 3-manifold by definition in [17]. (See Section 2.2.) The classification of such a 3-manifold in Corollary A in [17] implies that N is a generalized affine suspension of \mathbb{S}^2 , $\mathbb{R}P^2$, or a 2-hemisphere. To explain, N admits a total cross-section by Theorem B of Barbot [5]. This means that N and hence M are covered by $\mathbb{R}^3 - \{x\}$ or $H - \{x\}$ for the closed half-space H of \mathbb{R}^3 for $x \in \partial H$.

We now prove that when M is closed, the only case is the affine Hopf 3-manifold: M is a generalized affine suspension of a real projective 2-sphere or a real projective plane by the second item. In the first case, M has an infinite cyclic group as the deck transformation group acting on $\mathbb{R}^3 - \{O\}$. By Proposition A.2, M is an affine Hopf 3-manifold. In the second case, the double cover of M is an affine Hopf 3-manifold. An order two element k centralizes the infinite cyclic group since the generator fixes a unique point in \mathbb{R}^3 . $\pi_1(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. Also, k must act on a sphere in $\mathbb{R}^3 - \{O\}$ as an order two element, and, hence, $k = -I$. Thus, M is an affine Hopf 3-manifold.

When M is a generalized affine suspension over a 2-hemisphere, similar arguments apply to show that M is a half-Hopf manifold.

Any affine 3-manifold covered by an affine Hopf 3-manifold or a half-Hopf 3-manifold satisfies the premises of the theorem. Thus, it is an affine Hopf 3-manifold or a half-Hopf 3-manifold. □

Corollary 2.7. *Let M be a closed real projective 3-manifold. Suppose that M_J is a domain Ω in \mathbb{S}^3 with isolated boundary point x . Then M is projectively diffeomorphic to an affine Hopf 3-manifold.*

Proof. Let S be an ϵ -sphere, $0 < \epsilon < \pi/4$, in a neighborhood of x in Ω . A component U of $\mathbb{S}^3 - S$ contains x . Since M_J covers a compact

manifold, there exists a deck transformation g so $g(S) \subset U$ by Lemma 2.4. If g represented as a positive scalar times a unipotent matrix in any subspace V' of \mathbb{R}^4 with $x \in \mathbb{S}(V)$, then $\mathbb{S}(V') \cap S$ cannot go into $\mathbb{S}(V') \cap U$ under g . Thus, g acts on a subspace W of \mathbb{S}^3 disjoint from x and acts as an affine transformation on the affine space \mathbb{H}^o bounded by W . Proposition A.3 implies that x is an attracting fixed point of g on \mathbb{H}^o .

Now $g^i(S)$ is in the 3-ball B bounded by S for $i > 0$. Then $\bigcup_{j \in \mathbb{Z}} g^j(B)$ is projectively diffeomorphic to \mathbb{H}^o . Since $B - \{x\} \in \Omega$, we have $\mathbb{H}^o - \{x\} \subset \Omega$. Since $g^j(\partial B)$ geometrically converges to $\partial \mathbb{H}$ as $j \rightarrow -\infty$, and $S = \partial B$ is compact, we obtain $\Omega = \mathbb{H}^o - \{x\}$ by the proper-discontinuity of the action of $\langle g \rangle$. Theorem 2.6 shows that $\Omega/\langle g \rangle$ is an affine Hopf 3-manifold, a compact manifold. Therefore, M is finitely covered by an affine Hopf-3-manifold. Theorem 2.6 implies the result. \square

2.4. Convex concave decomposition of real projective 3-manifolds.

2.4.1. *Holonomy covers and Kuiper completions.* We recall results of [19]. However, we do this in more general regular cover M_J of M covering the holonomy cover $M_h = M/\ker h$. We take the universal cover \tilde{M} of a 3-manifold M . The existence of a real projective structure on M gives us

- an immersion $\mathbf{dev} : \tilde{M} \rightarrow \mathbb{R}P^3$, called a *developing map* and
- a homomorphism $h : \pi_1(M) \rightarrow \mathrm{PGL}(4, \mathbb{R})$, called a *holonomy homomorphism*

satisfying $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for each $\gamma \in \pi_1(M)$.

We will lift \mathbf{dev} to a map $\mathbf{dev}' : \tilde{M} \rightarrow \mathbb{S}^3$ and h lifts to $\pi_1(M) \rightarrow \mathrm{SL}_{\pm}(4, \mathbb{R})$ so that \mathbf{dev}' is an equivariant immersion. There exists

- a well-defined immersion $\mathbf{dev}' : \tilde{M} \rightarrow \mathbb{S}^3$ and
- a homomorphism $h' : \pi_1(M) \rightarrow \mathrm{SL}_{\pm}(4, \mathbb{R})$

so that $\mathbf{dev}' \circ g = h'(g) \circ \mathbf{dev}'$ for each deck transformation g of \tilde{M} .

We will now denote \mathbf{dev}' by \mathbf{dev} and h' by h for convenience for a given real projective 3-manifold M . (\mathbf{dev}, h) is determined only up to

$$(\mathbf{dev}, h(\cdot)) \mapsto (g \circ \mathbf{dev}, gh(\cdot)g^{-1})$$

for $g \in \mathrm{SL}_{\pm}(4, \mathbb{R})$.

Let K_h be the kernel of $h : \pi_1(M) \rightarrow \mathrm{SL}_{\pm}(4, \mathbb{R})$, independent of the choice of h . Let $M_h := \tilde{M}/K_h$ a so-called holonomy cover. Then \mathbf{dev} induces an immersion $\mathbf{dev}_h : M_h \rightarrow \mathbb{S}^3$. The deck transformation group Γ_h of M_h is isomorphic to $\pi_1(M)/K_h$. There is a homomorphism $h_{K_h} : \Gamma_h \rightarrow \mathrm{SL}_{\pm}(4, \mathbb{R})$ induced by h .

For a normal subgroup J of $\pi_1(M)$ with $J \subset K_h$, we define $M_J := \tilde{M}/J$. We let $p_J : M_J \rightarrow M$ denote the covering map. We obtain

- an immersion $\mathbf{dev}_J : M_J \rightarrow \mathbb{S}^3$, also called a *developing map* and
- a homomorphism $h_J : \pi_1(M)/J \rightarrow \mathrm{SL}_\pm(4, \mathbb{R})$, also called a *holonomy homomorphism*

satisfying

$$\mathbf{dev}_J \circ \gamma = h_J(\gamma) \circ \mathbf{dev}_J \text{ for } \gamma \in \Gamma_J.$$

The deck transformation group Γ_J of M_J is isomorphic to $\pi_1(M)/J$.

We recall the elementary fact that the regularity of the holonomy cover is preserved under restricting to submanifolds. (Hence, manifolds we will use here are regular covers of some submanifolds.)

Lemma 2.8. *For any connected submanifold N of M , let N_J denote a component of its inverse image in M_J . Then $p_J|_{N_J} : N_J \rightarrow N$ is a regular covering map also and the deck transformation group equals the subgroup Γ_{J,N_J} of Γ_J acting on N_J . For the developing map, we have $\mathbf{dev}_{J,N_J} = \mathbf{dev}_J|_{N_J}$ and for the corresponding holonomy homomorphism $h_{N_J} = h_J|_{\Gamma_{J,N_J}}$.*

The proof is elementary. We remark that even if $J = \{1\}$, N_J may not be a universal cover of N . (In theory, we could compute the deck transformation group.)

The immersion \mathbf{dev}_J induces a Riemannian μ -metric on M_J from the standard Riemannian metric μ on \mathbb{S}^3 . This gives us a path-metric to be denoted by \mathbf{d} on M_J . (More precisely \mathbf{d}_J but we omit J here.) Recall from [19], the Cauchy completion \check{M}_J of M_J with this path-metric is called the *Kuiper completion* of M_J . (This metric is quasi-isometrically defined by \mathbf{dev}_J and hence the topology is independent of the choice of \mathbf{dev}_J .) The *ideal set* is $M_{J,\infty} := \check{M}_J - M_J$, which is in general not empty. The immersion \mathbf{dev}_J extends to a continuous map. We use \mathbf{dev}_J as the notation for the extended map as well. If M is an affine 3-manifold, we define M as a real projective 3-manifold and \check{M}_J as above. Γ_J acts on M_J and $M_{J,\infty}$ possibly with fixed points in $M_{J,\infty}$.

- When $J = K_h$, we write \check{M}_h for \check{M}_{K_h} and $M_{h,\infty}$ for $M_{K_h,\infty}$.
- When $J = \{1\}$, we write \check{M} for $\check{M}_{\{1\}}$ and M_∞ for $M_{\{1\},\infty}$.

For a compact convex subset K of \check{M} so that $\mathbf{dev}_J|_K$ is an imbedding, we define ∂K to be the subset corresponding to $\partial \mathbf{dev}_J(K)$. If $\mathbf{dev}_J(K)$ is a compact convex domain in a subspace of \mathbb{S}^3 , then we define K° as the subset of K that is the inverse image of the manifold interior of $\mathbf{dev}_J(K)$. An *i-hemisphere* in \check{M}_J is a compact subset H

so that $\mathbf{dev}_J|_H$ is an imbedding to a i -hemisphere, $1 \leq i \leq 3$. A 3-*bihedron* in \check{M}_J is a compact subset B so that $\mathbf{dev}_J|_B$ is an imbedding to a compact convex set K so that ∂K is the union of two 2-hemispheres with the identical boundary great circle.

2.4.2. 2-convexity and covers. A *tetrahedron* or 3-simplex is a convex hull of four points in general position in an affine space \mathbb{R}^3 . A real projective 3-manifold M is *2-convex* if every projective map $f : T^\circ \cup F_1 \cup F_2 \cup F_3 \rightarrow M$ for a tetrahedron T with faces $F_i, i = 0, 1, 2, 3$, extends to one from T . (These definitions extend to affine manifolds considered as real projective manifolds.)

A *tetrahedron* in \check{M}_J is a compact subset T so that $\mathbf{dev}_J|_T$ is an imbedding to a tetrahedron in an affine space in \mathbb{S}^3 . A *face* of T is a corresponding subset of $\mathbf{dev}_J(T)$.

The following shows that the notion of 2-convexity can be characterized independent of J .

Proposition 2.9. *M is 2-convex if and only if for every tetrahedron T in \check{M}_J with faces $F_i, i = 0, 1, 2, 3$, such that $T^\circ \cup F_1 \cup F_2 \cup F_3 \subset M_J$, T is a subset of \check{M}_J .*

Proof. See Proposition 4.2 of [19] for the version for \check{M}_h . For general \check{M}_J , we have a covering map $M_J \rightarrow M_h$ is distance nonincreasing for the metrics \mathbf{d}_J and \mathbf{d}_{K_h} . It extends to a map $\check{M}_J \rightarrow \check{M}_h$. For any tetrahedron T in \check{M}_J , it is clear that the map is an isometry. Conversely, for a tetrahedron T in \check{M}_h , there exists a tetrahedron $T' \subset \check{M}_J$ mapping isometrically to T . \square

2.4.3. Crescents and two-faced submanifolds. We will now be discussing crescents in \check{M}_J . In [19], we defined these for \check{M}_h only. However, the theory will pass to \check{M}_J since M_J has trivial holonomy. If M is not 2-convex, then \check{M}_J contains a crescent by Theorem 1.2.

A *hemispherical 3-crescent* is a 3-hemisphere H in \check{M}_J so that a 2-hemisphere in ∂H is a subset of the ideal set $M_{J,\infty}$. We define α_R for a hemispherical 3-crescent R to be the union of all open 2-hemispheres in $\partial R \cap M_{J,\infty}$. We define $I_R = \partial R - \alpha_R$.

By Proposition 6.2 of [19] or by its \check{M}_J -version, given two hemispherical 3-crescents R and S in \check{M}_J , we have

- $R \cap S \cap M_J = \emptyset$,
- $R = S$, or
- $R \cap S \cap M_J$ is a union of common components of $I_R \cap M_J$ and $I_S \cap M_J$.

The components of $I_R \cap M_J$ as in the last case are called *copied components* of $I_R \cap M_J$. The union of all copied components in M_J , a *pre-two-faced submanifold of type I*, is totally geodesic and covers a compact imbedded totally geodesic 2-dimensional submanifold in M_J^o by Proposition 6.4 of [19]. The submanifold is called the *two-faced submanifold of type I* (arising from hemispherical 3-crescents). (It is possible that the needed results of [19] are true when the manifold-boundary ∂M is convex. This is not proved there.) Note it is possible that the two-faced submanifold of type I may be empty, i.e., does not exist at all.

The *splitting along* a submanifold A is given by the Cauchy completion M^s of $M - A$ of the path metric obtained by using an ordinary Riemannian metric on M and restricting to $M - A$. (Note this is not the Kuiper completion since we use the metric on M .)

Hypothesis 2.10. *We assume that a bihedral 3-crescent in \check{M}_J is defined if there is no hemispherical 3-crescent.*

In other words, we shall talk about bihedral 3-crescent when \check{M}_J has no hemispherical 3-crescent.

A *bihedral 3-crescent* is a 3-bihedron B in \check{M}_J so that a 2-hemisphere in ∂B is a subset of $M_{J,\infty}$. (We require that these are not contained in a hemispherical 3-crescent.) For a bihedral 3-crescent R , we define α_R as the open 2-hemisphere in $\partial R \cap M_{J,\infty}$. We define $I_R := \partial R - \alpha_R$, a 2-hemisphere. For a 3-crescent R , we define the interior of R as $R^o = R - I_R - \alpha_R$.

We say that two 3-crescents R and S *overlap* if $R^o \cap S \neq \emptyset$, or equivalently $R^o \cap S^o \neq \emptyset$. We say that $R \sim S$ if there exists a sequence of 3-crescents $R_1 = R, R_2, \dots, R_n = S$ where $R_i \cap R_{i+1}^o \neq \emptyset$ for $i = 1, \dots, n-1$.

We say that two bihedral 3-crescents R and S intersect *transversally* if

- $I_S \cap I_R$ is a segment with end points in ∂I_S and ∂I_R ,
- $I_S \cap R$ is the closure of a component of $I_S - I_R$
- $R \cap S$ is the closure of a component of $R - I_S$

In this case, $\alpha_S \cup \alpha_R$ is a union of two open 2-hemispheres meeting at an open convex disk $\alpha_S \cap \alpha_R$. Thus, they *extend* each other. (See Chapter 5 of [19].)

Proposition 2.11. *We assume as in Theorem 1.3. Suppose that two bihedral crescent R and S in \check{M}_J overlap. Then R and S either intersect transversally or $R \subset S$ or $S \subset R$. Moreover, $\mathbf{dev}_J|R \cup S$ is a*

homeomorphism to its image $\mathbf{dev}_J(R) \cup \mathbf{dev}_J(S)$ where $\mathbf{dev}_J(\alpha_R)$ and $\mathbf{dev}_J(\alpha_S)$ are 2-hemispheres in the boundary of a 3-hemisphere H .

Proof. This is a restatement of Theorem 5.4 and Corollary 5.8 of [19]. \square

From now on, assume that there is no hemispherical crescent in \check{M}_J . We define as in Chapter 7 of [19]

$$(2.1) \quad \begin{aligned} \Lambda(R) &:= \bigcup_{S \sim R} S, & \delta_\infty \Lambda(R) &:= \bigcup_{S \sim R} \alpha_S, \\ \Lambda_1(R) &:= \bigcup_{S \sim R} (S - I_S), & \delta_\infty \Lambda_1(R) &:= \delta_\infty \Lambda(R). \end{aligned}$$

We showed in Chapter 7 of [19] $\mathbf{dev}_J|_{\Lambda(R)}$ maps into a 3-hemisphere H and $\mathbf{dev}_J|_{\delta_\infty \Lambda(R)}$ is an immersion to ∂H (see also Corollary 5.8 of [19]).

Given a subset A of \check{M}_J , we define $\text{int}A$ to be the interior of A in \check{M}_J . We define $\text{bd}A$ to be the topological boundary of A in \check{M}_J . By Lemma 7.4 of [19], we have three possibilities:

- if $\text{int}\Lambda(R) \cap \Lambda(S) \cap M_J \neq \emptyset$ for two bihedral 3-crescents R and S , then $\Lambda(R) = \Lambda(S)$;
- $\Lambda(R) \cap \Lambda(S) \cap M_J = \emptyset$;
- $\Lambda(R) \cap \Lambda(S) \cap M_J \subset \text{bd}\Lambda(R) \cap M_J \cap \text{bd}\Lambda(S) \cap M_J$.

In the third case, the intersection is a union of common components of $\text{bd}\Lambda(R) \cap M_J$ and $\text{bd}\Lambda(S) \cap M_J$. We call such components *copied components*. These are totally geodesic and properly imbedded in M_J . The union of all copied components in M_J , a *pre-two-faced submanifold of type II*, covers a compact imbedded totally geodesic 2-dimensional submanifold in M° by Proposition 7.6 of [19]. The submanifold is called a *two-faced submanifold of type II* (arising from bihedral 3-crescents).

A *two-faced submanifold* is a two-faced submanifold of type I or type II.

2.4.4. Concave affine manifolds after splitting. Let M^s denote the 3-manifold obtained from M by splitting along the union of the two-faced submanifolds of type I. (Of course, we do not split if there are no two-faced submanifolds of type I and $M^s = M$.) A cover M_J^s of M^s can be obtained by splitting along the preimage of the union of the two-faced submanifold of type I in M_J and taking a component for every component of M^s and taking the union of these. (See Chapter 8 of [19].) For each component A of M_J^s , let Γ_A denote the subgroup of Γ_J acting on A° . Then Γ_A extends to a deck transformation group of A .

We define Γ_J^s the product group

$$\prod_{A \in \mathcal{C}} \Gamma_A \text{ for the set } \mathcal{C} \text{ of chosen components in } M_J^s.$$

Again M_J^s has a developing map $\mathbf{dev}_J^s : M_J^s \rightarrow \mathbb{S}^3$, an immersion, and $M_J^s \rightarrow M^s$ is a regular cover with the deck transformation group Γ_J^s . There is a map $M_J^s \rightarrow \check{M}$ by identifying along the splitting submanifolds. We can easily see that the [Kuiper completion](#) \check{M}_J^s contains the hemispherical 3-crescent if and only if \check{M}_J does. Also, the set of hemispherical 3-crescents of \check{M}_J^s maps in a one-to-one manner to the set of those in \check{M}_J by taking the interior of the hemispherical 3-crescent in \check{M}_J^s and sending it to \check{M}_J and taking the closure. (See Chapter 8 of [\[19\]](#)). Now \check{M}_J^s does not have any copied components. The map is onto up to the action of Γ_J^s .

Definition 2.12. *A compact real projective manifold with totally geodesic boundary covered by $R \cap M_J^s$ a hemispherical 3-crescent R is said to be a concave affine manifold of type I in M^s .*

Let \mathcal{H} be the set of all hemispherical crescents in M_J^s . The union $\bigcup_{R \in \mathcal{H}} R \cap M_J^s$ covers a finite union K of mutually disjoint concave affine manifolds of type I in M^s . Then $M^s - K^\circ = M_1$ is a compact real projective manifold with convex boundary. The cover $M_{1,J}$ of M_1 is \check{M}_J^s with all hemispherical 3-crescents removed from it. Then $\check{M}_{1,J}$ has no hemispherical 3-crescent. (We proved this in pages 80-81 of [\[19\]](#).)

Now, we look at M_1 only. We can easily see that the Kuiper completion $\check{M}_{1,J}$ contains the bihedral 3-crescent if and only if $\check{M}_{1,J}$ does. We split M_1 along the two-faced submanifold of type II if it exists. Let M_1^s denote the result of the splitting. Also, the set of bihedral 3-crescents of $\check{M}_{1,J}$ maps in a one-to-one manner to the set of those in $\check{M}_{1,J}^s$ by taking the interior of the bihedral 3-crescent and sending it to \check{M} and taking the closure. (See Chapter 8 of [\[19\]](#)). Now \check{M}_J^s does not have any copied components. For a bihedral crescent R in \check{M}_J^s , $\Lambda(R) \cap M_J^s$ covers a compact 3-manifold with concave boundary in M_J^s . (We prove this in pages 81-82 of [\[19\]](#).)

Definition 2.13. *Let R be a bihedral crescent in \check{M}_J . Suppose that \check{M}_J does not contain a hemispherical crescent. (See Hypothesis [2.10](#).) If $\Lambda(R) \cap M_J$ covers a compact real projective submanifold N , then N is called a concave affine manifold of type II.*

A concave affine manifold of type I or II is called a *concave affine manifold*.

(See Chapter 8 of [19] as a reference of results stated here.)

2.4.5. *Covering maps and crescents.* Let us clarify the relationship between the crescents in \check{M}_h and \check{M}_J .

Proposition 2.14. *Let M be a compact real projective 3-manifold with convex boundary. Suppose that M is not complete affine or bihedral. Let M_J be a regular cover of M_h with covering map $p_{J,h}$ for a subgroup J , $J \subset K_h$. Then the following hold:*

- Any hemispherical 3-crescent R in \check{M}_J maps to one in \check{M}_h by taking R° and sending it to M_h and taking its closure. Conversely, any hemispherical 3-crescent R' is obtained by such a procedure.
- The [pre-two-faced submanifold](#) in M_J arising from hemispherical 3-crescents covers one in M_h by $p_{J,h}$.
- A bihedral 3-crescent in \check{M}_J maps to one in \check{M}_h by the same procedure and the surjectivity is also true.
- The [pre-two-faced submanifold](#) in M_J arising from bihedral 3-crescents covers one in M_h by $p_{J,h}$.
- Let N_J be a component of the inverse image of a compact 3-manifold N in M . Let \check{N}_J denote the Kuiper completion of N_J using $\mathbf{dev}_J|_{N_J}$. The inclusion map $N_J \rightarrow M_J$ extends to the map $\check{N}_J \rightarrow \check{M}_J$. Then a hemispheric or bihedral 3-crescent R in \check{N}_J maps to one in \check{M}_J .

Proof. The map $p_{J,h} : M_J \rightarrow M_h$ is distance non-increasing since any path in M_J measured by the \mathbf{d} -metric goes to one in M_h of the same \mathbf{d} -length. Let $\hat{p}_{J,h}$ denote the obvious extension $\check{M}_J \rightarrow \check{M}_h$. An [open hemisphere](#) U in M_J maps injectively to an open hemisphere in M_h since $\mathbf{dev}_J|_U$ is injective and $\mathbf{dev}_h \circ p_{J,h} = \mathbf{dev}_J$.

Given an open 3-hemisphere H in M_J , $p_{J,h}|_H$ is a homeomorphism to its image in M_h since $\mathbf{dev}_h \circ p_{J,h}|_H = \mathbf{dev}_J|_H$ is a homeomorphism to its image in \mathbb{S}^3 by [definition](#). For any hemispherical 3-crescent R in \check{M}_J , the image R' in M_h has a 3-hemisphere H as the interior. The closure of H must be a 3-hemisphere by the consideration of the \mathbf{d} -metric, and hence equals R' . The subset $M_J \cap R$ maps to $M_h \cap R'$ as a covering map. It must be a homeomorphism since $R^\circ \rightarrow H$ is one. The complement of $M_h \cap R'$ in R' is in $M_{J,\infty}$. Hence, this proves that R' is a hemispherical 3-crescent.

The rest follow similarly. \square

3. CONCAVE AFFINE 3-MANIFOLDS

In this section, we will prove Theorems 3.2, 3.7 and 3.8. The first one shows that the compressible two-faced submanifolds cannot happen in general. In the second and third ones, we showed that a concave affine manifold with compressible boundary contains a [toral \$\pi\$ -submanifold](#).

Given an imbedded surface Σ in a 3-manifold M that is either on the boundary of M or is two-sided, Σ is *incompressible* into M if $\pi_1(\Sigma)$ injects into $\pi_1(M)$. Otherwise, Σ is said to be *compressible*. A simple closed curve in Σ is *essential* if it is not null-homotopic in Σ . A compressible surface always has an essential simple closed curve that is the boundary of an imbedded disk by Dehn's Lemma.

3.1. A concave affine manifold has no sphere boundary component.

Lemma 3.1. *Let N be a concave affine manifold of type I or II. Then no component of ∂N is covered by a sphere.*

Proof. If N is a concave affine manifold of type I, then N is covered by $\tilde{N} = R^\circ \cup I_R \cap N_J$ for a hemispherical crescent R . Since $I_R \cap N_J$ is an open surface in I_R , the conclusion follows.

Suppose now that there is no hemispherical crescent in M_J . (See Hypothesis 2.10.) Then N is covered by $\Lambda(R) \cap M_J$ for a bihedral crescent R . Suppose that a component A of $\text{bd}\Lambda(R) \cap M_J$ is a sphere. We know that A maps to a convex surface in $M - N^\circ$ under the covering map. If A is totally geodesic, then A is tangent to $I_S \cap M_J$ for a crescent S in $\Lambda(R)$. Hence, A is a subset of $I_S \cap M_J$, each component of which is not compact. This is a contradiction.

Suppose that A is foliated by geodesics. Then the geodesic leaf must be in I_S for a crescent S and must end at some point. This implies that A is not compact, a contradiction.

Hence, there exists a point y where A is convex but not totally geodesic as a boundary point of $M_J - \Lambda(R)$ and A is also not foliated by geodesic segments in a neighborhood of y in A . This contradicts Theorem B.1. □

3.2. Compressible two-faced submanifolds.

Theorem 3.2. *Suppose that a compact real projective 3-manifold M with empty or convex boundary and M is not complete affine or bihedral. Suppose that M contains a [two-faced submanifold](#) S in M . Then either S is incompressible in M or M is an affine Hopf 3-manifold.*

This implies that two-faced submanifolds are incompressible unless M is an affine Hopf 3-manifold.

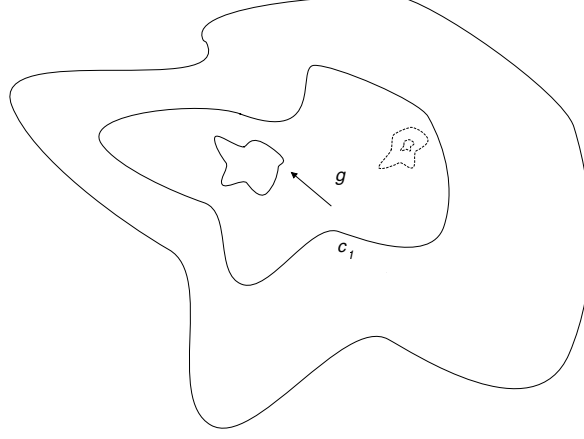


FIGURE 1. There must be an image of the curve c_1 inside the disk bounded by c_1 .

(I) Let A_1 denote a component of a two-faced submanifold of type I. Suppose that A_1 is covered by a component \tilde{A}_1 of $I_R \cap M_J$ for a hemispherical 3-crescent R . If \tilde{A}_1 is simply connected, then A_1 is incompressible in M and we are done here. Let Γ_1 denote the deck transformation group of \tilde{A}_1 in Γ_J so that \tilde{A}_1/Γ_1 is compact and diffeomorphic to A_1 .

Now assume that A_1 is compressible in M . By Dehn's lemma, A_1 has an essential simple closed curve bounding a disk in M . We lift the closed curve to \tilde{A}_1 , and \tilde{A}_1 contains a simple closed curve c_1 not bounding a disk in \tilde{A}_1 . We may assume that c_1 imbeds to a simple closed curve in the two-faced submanifold by taking a finite cover of M if necessary. (A preimage of a two-faced submanifold is still one by Proposition 8.13 of [19].) Thus, $\{g(c_1) | g \in \Gamma_1\}$ is a collection of mutually disjoint curves. Then c_1 bounds a compact disk D_1 in I_R^o not contained in \tilde{A}_1 , and $D_1 \cap \tilde{A}_1$ contains a component D'_1 of $\tilde{A}_1 - c_1$. Since \tilde{A}_1 regularly covers a compact submanifold, there exists a deck transformation g acting on \tilde{A}_1 so that

$$g(c_1) \subset \tilde{A}_1 \cap D_1$$

by Lemma 2.4. We can assume that g is orientation preserving by taking a finite cover of M if necessary. Since $g(R)^o \cup D_1$ and $R^o \cup D_1$

are one-sided neighborhoods of \tilde{D}_1 in M_J , it follows that $g(R)$ and R^o meet. We obtain $g(R) = R$ by Theorem 5.1 of [19]. It follows

$$g(D_1) \subset D_1^o \text{ and } g(I_R) = I_R.$$

Since c_1 is separating in I_R^o , it follows that $g^i(D_1) \subset D_1^o$ for every $i > 0$.

By the Brouwer fixed point theorem, g fixes a point x in I_R^o . Proposition A.3 implies that x is a global attracting fixed point of the contracting affine transformation $g|_{I_R^o}$. Since $g^{-1}|_{I_R^o}$ is an expanding linear map, the set $\{g^i(c_1)\}$ is a collection of curves forming a sequence geometrically converging to the great 2-sphere ∂I_R as $i \rightarrow -\infty$. (See Proposition 1.2.8 of [35].) By Propositions 6.4 and 7.6 in [19], each copied component \tilde{A}_j of $I_R \cap M_J$ covers a compact submanifold in M_J . Non-copied component \tilde{A}_j of $I_R \cap M_J$ is mapped homeomorphically to a compact surface again: $\{g(\tilde{A}_j) | g \in \Gamma_J\}$ is a mutually disjoint collection of closed sets and is locally finite as in a part of the proof of Theorem 9.3 [19] since it is a boundary component of $I_R \cap M_J$.

For each integer k , $g^k(c_1)$ and $g^{k+1}(c_1)$ bound an annulus \mathcal{A}_k and are disjoint from \tilde{A}_j , $j \neq 1$. By the eigenvalue condition $\bigcup_{k \in \mathbb{Z}} \text{Cl}(\mathcal{A}_k) = I_R^o - \{x\}$. Thus, for some k , we obtain $\tilde{A}_j \subset \mathcal{A}_k$ if $j \neq 1$. Hence, \tilde{A}_j for $j \neq 1$ is a precompact subset of I_R^o . By Lemma 3.3, every \tilde{A}_j , $j \neq 1$, cannot cover a compact submanifold. Therefore, we obtain that \tilde{A}_1 is a unique component of $I_R \cap M_J$.

Since

$$\tilde{A}_1 = I_R \cap M_J = I_S \cap M_J$$

is a pre-two-faced submanifold, \tilde{A}_1 covers a compact 2-manifold. By the classification of affine 2-manifolds (see [8]), we obtain

$$\tilde{A}_1 = I_R - \{x\} \text{ as } \tilde{A}_1 \supset c_1.$$

Since \tilde{A}_1 covers a two-faced submanifold, \tilde{A}_1 is a component of $I_S \cap M_J$ for a hemispherical 3-crescent S where

$$R \cap S \cap M_J = \tilde{A}_1.$$

Since $\text{Cl}(\alpha_S) \cup \text{Cl}(\alpha_R) \subset M_\infty$ bounds the compact domain $R \cup S$ in \check{M} , we obtain $R^o \cup S^o \cup \tilde{A}_1 = M_J$. Since R is a 3-hemisphere, we have

$$g(R) = R \text{ or } g(R) = S.$$

Thus, the deck transformation group $\langle g \rangle$ acts on \tilde{A}_1 . However, \tilde{A}_1/Γ_1 is a 2-dimensional closed surface while M_J/Γ_J is a closed 3-manifold. We can assume that these are orientable by taking finite coverings. Since

$$M_J = R^o \cup S^o \cup (I_R^o - \{x\}) \cong \mathbb{S}^2 \times \mathbb{R},$$

it follows that M is finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ considering $\mathbb{S}^2 \times I / \sim$ by Lemma 3.4. $\mathbf{dev}_J|R^\circ \cup I_R^\circ - \{x\}$ and $\mathbf{dev}_J|S^\circ \cup I_S^\circ - \{x\}$ are homeomorphisms to their images. Thus, $\mathbf{dev}_J|M_J$ is a homeomorphism to the image

$$\mathbf{dev}_J(R)^\circ \cup \mathbf{dev}_J(S)^\circ \cup \mathbf{dev}_J(I_R^\circ) - \mathbf{dev}_J(x).$$

Since $\mathbf{dev}_J(x)$ is an isolated boundary point, by Corollary 2.7, we are finished in this case.

(II) Let A_1 denote a component of a two-faced submanifold of type II in M that is compressible. Now, we assume that \tilde{M}_J has no hemispherical crescent. (See Hypothesis 2.10.) Then its cover \tilde{A}_1 is a component of $I_R \cap M_J$ containing a noncontractible closed curve for a bihedral crescent R . Using the arguments are very similar to the case (I) and a part of the proof of Theorem 9.3 of [19], we obtain that $\tilde{A}_1 = I_R^\circ - \{x\}$ for a bihedral 3-crescent R .

Since \tilde{A}_1 is in a pre-two-sided submanifold, we obtain that $I_R \subset I_S$ for another bihedral 3-crescent S so that $S^\circ \cap R^\circ = \emptyset$. It follows that

$$I_R^\circ - \{x\} = I_S^\circ - \{x\} \text{ and hence } I_R = I_S.$$

Since $\text{Cl}(\alpha_R) \cup \text{Cl}(\alpha_S) \subset M_{J,\infty}$ forms the boundary of $R \cup S$, and M_J is disjoint from it,

$$M_J = R^\circ \cup S^\circ \cup I_R^\circ - \{x\}$$

holds. Thus, M_J is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. It follows as above that M is finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ by Lemma 3.4. Again, we use the lifted developing map $\mathbf{dev}_J : M_J \rightarrow \mathbb{S}^3$ as above. Since $\mathbf{dev}_J(x)$ is an isolated boundary point, Corollary 2.7 implies the result in this case.

Lemma 3.3. *Let Ω_1 be an open surface in M_J with $\mathbf{dev}_J(\Omega_1)$ imbedded in a properly imbedded hypersurface S and bounded in an affine space $A = H^\circ$ for a 2- or 3-hemisphere H . Suppose that a discrete group $G \subset \Gamma_J$ acts properly discontinuously and freely on Ω_1 , and $h_J|G$ is injective. Moreover, G acts on H . Then Ω_1/G is noncompact.*

Proof. Suppose that there exists G so that Ω_1/G is compact. Since G acts on H , G acts as a group of affine transformations on the affine 2- or 3-space H° . Let F be the compact fundamental domain of Ω_1 . The closure $\text{Cl}(\mathbf{dev}_J(\Omega_1))$ is a compact bounded subset of A . The convex hull C_1 of $\text{Cl}(\mathbf{dev}_J(\Omega_1))$ is a bounded subset of A also, and G acts on it and its center of mass m , and hence $h_J(G)$ is a group of affine bounded transformations fixing m . By choosing an invariant Euclidean metric on A , the group $h_J(G)$ acts isometrically. Since Ω is open, there exists a sequence $\{y_i\}$ exiting all compact sets in Ω eventually. There

exists $g_i \in G$ such that $g_i(y_i) \in F$. By taking a subsequence, we may assume $\mathbf{dev}_J(y_i) \rightarrow y \in S$ and y is in the boundary of $\mathbf{dev}_J(\Omega)$, i.e., $y \notin \mathbf{dev}_J(\Omega)$. Then $g_i^{-1}(F) \ni y_i$. Let U be an open neighborhood of F in Ω . Since $\mathbf{dev}_J(y_i) \rightarrow y$, $h_J(g_i^{-1})$ is an isometry group fixing m , and S , $S \ni y$, is properly imbedded, it follows that

$$\mathbf{dev}_J(\Omega) \supset \mathbf{dev}_J(g_i^{-1}(U^o)) = h_J(g_i^{-1})(\mathbf{dev}_J(U^o)) \ni y \text{ for sufficiently large } i,$$

which is a contradiction. \square

Let $\text{Diff}(K)$ be the group of diffeomorphisms of K with the usual C^r -topology and $\text{Diff}_0(K)$ the identity component of this group. We define the mapping class group $\text{Mod}(K)$ of a manifold K to be the group $\text{Diff}(K)/\text{Diff}_0(K)$.

Since $\text{Mod}(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$ is a classical work of Smale [39], there exist only two homeomorphism types of \mathbb{S}^2 -bundle over \mathbb{S}^1 . If M' is orientable, then M' is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. If not, M' is a non-orientable \mathbb{S}^2 -bundle over \mathbb{S}^1 .

Lemma 3.4. *Suppose that $\tilde{N} = K \times \mathbb{R}$ for a compact manifold K covers a compact manifold N as a regular cover. Suppose that $\text{Mod}(K)$ is finite. Then N is finitely covered by $K \times \mathbb{S}^1$.*

Proof. There exists a deck transformation g so that $K \times \{0\} \cap g(K \times \{0\}) = \emptyset$ since the deck transformation group acts properly discontinuously. $K \times \{0\}$ is an incompressible surface in $K \times \mathbb{R}$. Now $K \times \{0\} \cup g(K \times \{0\})$ bounds a compact submanifold N_1 . By incompressibility, $\pi_1(K) \rightarrow \pi_1(N_1)$ is injective. Now, $\pi_1(K \times \mathbb{R})$ must be amalgamation of infinite copies of $\pi_1(N_1)$. Hence, $\pi_1(K) \rightarrow \pi_1(N_1)$ is an isomorphism. Thus, N_1 is homeomorphic to $K \times I$ for an interval I . (See Chapter 10 of [30].) N is finitely covered by a bundle N_2 over \mathbb{S}^1 with fiber diffeomorphic to K . The bundle N_2 is diffeomorphic to

$$K \times I / \sim \text{ where } (x, 0) \sim (g(x), 1)$$

where $g : K \rightarrow K$ is a diffeomorphism. Since $\text{Mod}(K)$ is finite, a power g^i is isotopic to the identity for an integer $i \geq 1$. Thus, an i -fold cover of N_2 is diffeomorphic to $K \times \mathbb{S}^1$. \square

3.3. Concave affine manifolds and toral π -manifolds.

Definition 3.5. *Suppose that \tilde{M}_J contains two crescents S_1 and S_2 so that $I_{S_1} \cap M_J$ and $I_{S_2} \cap M_J$ intersect and are tangent but $\mathbf{dev}_J(S_1)^o \cap \mathbf{dev}_J(S_2)^o = \emptyset$. In this case S_1 and S_2 are said to be opposite.*

Definition 3.6. *Suppose that a real projective manifold M is not complete affine or bihedral and let M_J be a regular cover of the holonomy*

cover M_h of M . Assume that M has no two-sided submanifolds. Let R be a hemispheric 3-crescent with $I_R \cap M_J = I_R^\circ - \{x\}$ for $x \in I_R^\circ$. Then a compact submanifold P covered by $R \cap M_J$ is called a toral π -submanifold of type I.

Suppose that \check{M}_J has no hemispheric crescent. (See Hypothesis 2.10.) Given $\Lambda(R)$ for a bihedral crescent R , we define the set $C_{R,x}$ as follows: Suppose that for some $x \in \mathbb{S}^3$, we have

$$C_{R,x} := \{R' \mid R' \sim R, \exists g \in \Gamma_J, g(R) = R, h_J(g)(x) = x, \mathbf{dev}_J(I_{R'}^\circ) \ni x\} \neq \emptyset.$$

Let $\Lambda'(R)$ be $\bigcup_{R' \in C_{R,x}} R'$ whenever $C_{R,x}$ is not empty and

$$\delta_\infty \Lambda'(R) := \bigcup_{S \in C_{R,x}} \alpha_S.$$

Then $\Lambda'(R)$ develops into a 3-hemisphere H and $\delta_\infty \Lambda'(R)$ develops to an open disk in ∂H for a 3-hemisphere H by Chapter 7 of [19]. Suppose that $\Lambda'(R) \cap M_J$ covers a compact radiant affine 3-manifold P with compressible boundary. Then P is said to be a toral π -submanifold of type II.

A total π -submanifold of type I or II is called a *total π -submanifold*.

Theorem 3.7. *Let N be a concave affine 3-manifold with nonempty boundary ∂N in a compact real projective 3-manifold M with empty or convex boundary. Suppose that M is not complete affine or bihedral. Assume that M has no *two-faced submanifold of type I*. Let M_J be a regular cover M covering the holonomy cover of M . Suppose that N is a concave affine 3-manifold of type I with compressible boundary ∂N . Then one of the following holds:*

- M is an affine Hopf 3-manifold,
- N has a unique boundary component A compressible into N , and N is a toral π -submanifold P of type I.

Theorem 3.8. *Let N be a concave affine 3-manifold with nonempty boundary ∂N in a compact real projective 3-manifold M with empty or convex boundary. Suppose that M is not complete affine or bihedral. Let M_J be a regular cover M covering the holonomy cover of M . Suppose that \check{M}_J has no hemispherical 3-crescent. (See Hypothesis 2.10.) Assume that M has no two-faced submanifold of type II. Suppose that N is a concave affine 3-manifold of type II with compressible boundary ∂N . Then one of the following holds:*

- M is an affine Hopf 3-manifold,
- N has a unique boundary component A compressible into N , and N contains a toral π -submanifold P of type II. Furthermore, the following holds:

- Let $N_J \subset M_J$ be a component of the inverse image of N . The inverse image of P in N_J meets the interior of any 3-crescent in the Kuiper completion \check{N}_J of N_J . The fundamental group of N is virtually infinite-cyclic.
- Let R be a 3-crescent in $\text{Cl}(N_J)$ in \check{M}_h . Then R is a bihedral 3-crescent and $\mathbf{dev}_J|_{\Lambda_1(R)}$ is a homeomorphism to $H - K$ for a properly convex compact domain K in a 3-hemisphere H with $K \cap \partial H \neq \emptyset$.

To prove Theorems 3.7 and 3.8, we just need to study the case when N is a [concave affine 3-manifold](#) with compressible boundary. Let J be a normal subgroup of $\pi_1(M)$ contained in K_h . Let M_J denote a regular cover of M with developing map \mathbf{dev}_J and the holonomy homomorphism h_J . Let Γ_J denote the deck transformation group of M_J isomorphic to $\pi_1(M)/J$. h_J is not necessarily injective from Γ_J to $\text{SL}_{\pm}(4, \mathbb{R})$. The kernel equals K_h/J . Let N_J denote a component of the inverse image of N in M_J as in the premise.

Suppose that we have a bihedral 3-crescent R in \check{N}_J so that a deck transformation g acts on $R^o \cup I_R^o - \{x\} \subset N_J$. We call such a bihedral 3-crescent a [toral bihedral 3-crescent](#).

Now, we have no pre-two-faced submanifold by the premise of Theorems 3.7 and 3.8.

This proof is fairly long. To outline, we give the following:

- (I) Concave affine 3-manifolds of type I.
- (II) Concave affine 3-manifolds of type II.
 - (A) there exist three mutually [overlapping](#) bihedral 3-crescents in $\Lambda(R)$ for a bihedral crescent R in \check{M}_J .
 - (i) There is a pair of [opposite](#) bihedral 3-crescents in $\Lambda(R)$. By Lemma 3.10, M is covered by an affine Hopf 3-manifold finitely.
 - (ii) Otherwise, $\mathbf{dev}_J|_{\Lambda_1(R)}$ is a homeomorphism to $H - K$ for a properly convex domain K and a 3-hemisphere H containing K , and $\Lambda(R)$ contains a toral bihedral 3-crescent. Lemma 3.13 gives us a toral π -submanifold. We obtain this by three steps (a), (b), (c) below.
 - (B) Otherwise, all bihedral 3-crescents R have $\mathbf{dev}_J(I_R)$ containing a fixed pair of points q, q_- . Then $\Lambda(R)$ is a union of segments from q to q_- .
 - (i) A closed curve in a component A_1 of $\text{bd}\Lambda(R) \cap M_J$ bounds a disk in the union $A_{1,+}$ of lines from q to q_- passing A_1 . Here, the situation is similar to (A)(i) and we use Lemma 3.13.

- (ii) Otherwise, $A_{1,+}$ is an annulus. We show that this case does not happen.

After (II), we assume that there is no hemispherical 3-crescent in \check{M}_J . (See Hypothesis 2.10.)

(I) Let N be a concave affine 3-manifold of type I in M . Then $F \cap M_J$ covers N for a hemispherical 3-crescent F . Let Γ_N denote the subgroup of Γ_J acting on $F \cap M_J$ as the deck transformation group of the covering map to N .

Let \tilde{A}_1 denote a component of $I_F \cap M_J$. If it is simply connected, then A_1 is incompressible and we are finished for this component.

Suppose that \tilde{A}_1 is not simply connected. Then \tilde{A}_1 contains a simple close curve c_1 . As in the part (I) of the proof of Theorem 3.2, we can show using Lemma 3.3 that

$$\tilde{A} = I_F^\circ \cap M_J = I_F^\circ - \{x\}$$

holds. By Dehn's lemma, we can obtain a compressing disk in $F^\circ \cup I_F^\circ - \{x\}$ with boundary in $I_F^\circ - \{x\}$. Thus,

$$F \cap M_J = F^\circ \cup I_F^\circ - \{x\}$$

is homeomorphic to $D^2 \times \mathbb{R}$, and N is covered by $D^2 \times \mathbb{S}^1$ by Lemma 3.4.

Since the fundamental group of N acts on an annulus $I_F^\circ - \{x\}$ properly discontinuously and freely, Γ_N is virtually infinite-cyclic. Thus, $(I_F^\circ - \{x\})/\Gamma_N$ can only be a torus, a Klein bottle, an annulus, or a Möbius band.

(II) Now suppose that N is a concave affine 3-manifold of type II in M . we assume that there is no hemispherical 3-crescent in \check{M}_J . Then N is covered by $\Lambda(R) \cap M_J$ for a bihedral 3-crescent R in \check{M}_J . Let Γ_N denote the subgroup of Γ_J acting on $\Lambda(R) \cap M_J$ as the deck transformation group of the covering map to N . Recall that

$$\mathbf{dev}_J(\Lambda(R)) \subset H, \delta_\infty \Lambda(R) \subset \partial H$$

for a 3-hemisphere $H \subset \mathbb{S}^3$. (See Corollary 5.8 of [19].)

(A) Suppose that there exist three mutually overlapping bihedral 3-crescents R_1, R_2 , and R_3 with $\{I_{R_i} | i = 1, 2, 3\}$ in general position. By modifying the proofs of Lemma 11.1 and Proposition 11.1 of [17] for bihedral 3-crescents, which are not necessarily radial as in the paper, we obtain that

$$(3.1) \quad \begin{aligned} \mathbf{dev}_J : \Lambda_1(R) &\rightarrow H - K, \\ \mathbf{dev}_J|_{\Lambda_1(R) \cap M_J} : \Lambda_1(R) \cap M_J &\rightarrow H^\circ - K \end{aligned}$$

are homeomorphisms for a 3-hemisphere H and a compact convex set K . (For Lemma 11.1 and Proposition 11.1 of [17], we do not need I_R for each bihedral 3-crescent to contain the origin. There is a typo in the third line of the proof of Lemma 11.1 of [17]. We need to change $P_1 \cap L_1$ and $P_1 \cap L_2$ to $P_1 \cap L_2$ and $P_1 \cap L_3$ respectively.) The general position property of I_{R_i} , $i = 1, 2, 3$, implies that K is properly convex. Also, the equation (3.1) implies that $h_J|_{\Gamma_N}$ is injective.

Here, $\mathbf{dev}_J(\alpha_{R'}) \subset \partial H$ for $R' \sim R$. Now, $\text{bd}\Lambda_1(R) \cap M_J$ maps into $\text{bd}K$. Let K' denote the inverse image in $\text{bd}\Lambda_1(R)$ of K . $h_J(\Gamma_N)$ is an affine transformation group of H^o since it acts on an affine space H^o as a projective automorphism group.

Hypothesis 3.9. *We can have two possibilities:*

- (A)(i) *Suppose that there exist two opposite bihedral 3-crescents $S_1, S_2 \sim R$.*
- (A)(ii) *There are no such bihedral 3-crescents.*

(A)(i) At least one component A_1 of $I_{S_1}^o \cap M_J$ contains $I_{S_1}^o - K'$ for a properly convex compact set K' by equation (3.1).

Here, we will show that M is an affine Hopf 3-manifold. The following finishes (A)(i).

Lemma 3.10. *Suppose that there exist two bihedral 3-crescents S_1, S_2 in \check{M}_J so that $I_{S_1} \cap M_J$ and $I_{S_2} \cap M_J$ intersect and are tangent but $\mathbf{dev}_J(S_1)^o \cap \mathbf{dev}_J(S_2)^o = \emptyset$. Assume $S_1, S_2 \sim R$. Then there exists a unique component of $I_{S_i} \cap M_J$ equal to $I_{S_i}^o - \{x\}$ for a point x of $I_{S_i}^o$, $i = 1, 2$, and M is an affine Hopf 3-manifold.*

Proof. First, $I_{S_1} \cap M_J$ and $I_{S_2} \cap M_J$ meet at the union of their common components by geometry since such a component is totally geodesic and complete in M_J .

We will show that A_1 is a unique component of $I_{S_1}^o \cap M_J$: Suppose that there exists a component A_2 of $I_{S_1}^o \cap M_J$ different from A_1 . Then $A_2 \subset I_{S_1}^o \cap K'$ and is disjoint from A_1 .

We now show that

$$\{g(A_2) | g \in \Gamma_J\}$$

is a locally finite collection of disjoint closed sets following Section 3.5 of [20]: Suppose that $g_1(A_2)$ and $g_2(A_2)$ intersect. Then $g_2^{-1}g_1(S_1)$ intersects S_1^o or S_2^o and hence we have $g_2^{-1}g_1(S_1) \sim R$. This implies that $g_2^{-1}g_1(S_1^o) \subset \Lambda_1(R)$. Since $g_2^{-1}g_1(S_2^o) \subset \Lambda_1(R)$ also, we have that $g_2^{-1}g_1(A_2)$ is tangent to A_2 . Since they are both maximal totally geodesic hypersurfaces, we obtain $g_2^{-1}g_1(A_2) = A_2$. Hence $g_1(A_2) = g_2(A_2)$.

We show that $\{g(A_2) | g \in \Gamma_J\}$ is a locally finite collection of disjoint closed sets: Suppose that a sequence $\{p_i \in g_i(A_2) | i \in \mathbb{Z}_+\}$ converges to $p \in M_J$ with $g_i(A_2)$ are mutually distinct. Consider a convex open ball $B(p) \subset M_J$ of p . Since $g_i(A_2)$ is a properly imbedded open hypersurface in M_J , $g_i(S_1) \cap B(p)$ is one of components of $B(p) - g_i(A_2)$, which are one or two. Then since $\{g_i(A_2)\}$ are mutually disjoint, we obtain that

$$g_i(A_2) \cap B(p) \subset g_j(S_1^o) \cap B(p) \text{ or } g_j(A_2) \cap B(p) \subset g_i(S_1^o) \cap B(p)$$

for a fixed i and infinitely many j . Thus,

$$A_2 \cap B(p) \subset g_i^{-1}g_j(S_1^o) \text{ or } A_2 \cap B(p) \subset g_j^{-1}g_i(S_1^o).$$

Then

$$g_i^{-1}g_j(S_1) \sim S_1 \text{ or } g_j^{-1}g_i(S_1) \sim S_1$$

since $A_2 \subset S_1$. If $g_i^{-1}g_j(A_2)$ or $g_j^{-1}g_i(A_2)$ meets S^o for any bihedral 3-crescent S , $S \sim R$, then A_2 meets $g_i^{-1}g_j(S^o)$ or $g_j^{-1}g_i(S^o)$; but then A_2 cannot be in $\text{bd}\Lambda_1(R) \cap M_J$. This is a contradiction and $\{g_i(A_2)\}$ is a locally finite collection.

Therefore, A_2 covers a compact closed 2-manifold B_2 in M . Let Γ_2 denote the subgroup of deck transformations acting on A_2 in Γ_N .

$$\text{If } g(A_2) = A_2, g \in \Gamma_2, \text{ then } g(\Lambda(R)) = \Lambda(R)$$

since both one-sided neighborhoods of A_2 are in S_1^o and S_2^o . Since A_2 has a one-sided neighborhood S_1 , we obtain $g(S_1)^o \cap S_2 \neq \emptyset$. Recall that $\mathbf{dev}_J|_{\Lambda(R)}$ is a map into a 3-hemisphere H with $\delta_\infty \Lambda(R)$ going into ∂H . Since $\mathbf{dev}_J|_{\delta_\infty \Lambda(R)}$ immerses to ∂H for the 3-hemisphere, and $g(S_1) \cap S_1^o \neq \emptyset$,

- we obtain $g(S_1) = S_1$ by Theorem 5.4 in [19] and
- hence $g(I_{S_1}^o) = I_{S_1}^o$ for the affine space $I_{S_1}^o$ and $g \in \Gamma_2$.

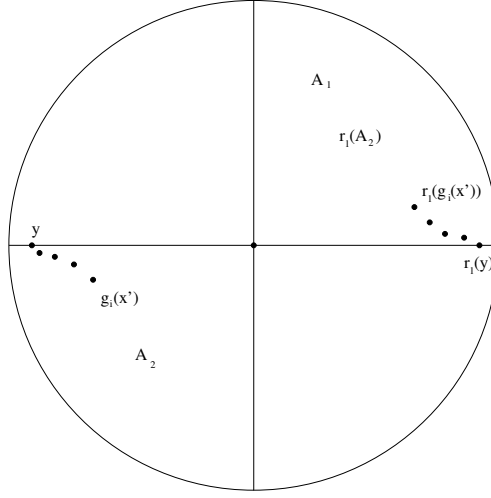
Thus, B_2 has an affine structure.

Since $A_2 \subset I_{S_1}^o \cap K'$ with properly convex $\mathbf{dev}_J(K') \subset K$, the classification of affine 2-manifolds (see [10]) implies that $\text{Cl}(A_2)$ is a properly convex triangle with a vertex $x \in I_{S_1}^o$. Hence Γ_2 fixes x and acts properly discontinuously and cocompactly on A_2 . We may assume that Γ_2 is abelian by taking a finite index cover.

Let E be an edge of the triangle A_2 with an endpoint x . From the properties of the action of a cocompact linear group acting on a proper cone, we find that there exists a sequence $g_i(x') \rightarrow y \in E^o$ for $x' \in A_2$ for a sequence $g_i \in \Gamma_2$ of mutually distinct elements.

We need an order two affine transformation r_1 of $I_{S_1}^o$ fixing x so that

- r_1 equals $-I$ on the affine space $I_{S_1}^o$ for an affine coordinate system but

FIGURE 2. The action of r_1 and the action of Γ_2 .

- r_1 is not necessarily in the deck transformation group

Let $A'_2 := r_1(A_2)$ in $I_{S_1}^o$. Here, $r_1 g_i = g_i r_1 | I_{S_1}^o$ since Γ_2 is abelian and fixes x . Then

$$g_i(r_1(x')) \rightarrow r_1(y) \in r_1(E)^o.$$

Since K is properly convex and $I_{S_1}^o \cap K' \subset \text{Cl}(A_2)$, we obtain

$$r_1(y) \in r_1(E)^o \subset I_{S_1}^o \cap M_J.$$

We can choose x' and y (far from x) so that $r_1(y) \in A_1 \subset M_J$. Now

$$r_1(y) \in A_1 \subset M_J, g_i(r_1(x')) \rightarrow r_1(y)$$

contradict the proper discontinuity of the action of the deck transformation group on $I_{S_1}^o - K'$. Hence, we conclude that A_1 is the only component of $I_{S_1} \cap M_J$ and $I_{S_2} \cap M_J$. (See Figure 2.)

As above for A_2 , we can show that A_1 covers a compact affine 2-manifold in M . By the classification of the affine 2-manifolds (see [8]),

$$A_1 = \begin{cases} I_{S_1}^o & \text{or} \\ I_{S_1}^o - \{x\}, x \in I_{S_1}^o. \end{cases}$$

In the first case, we obtain that

$$\text{dev}_J(\Lambda_1(R) \cap M_J) = H^o \text{ and}$$

$$\partial H = \text{Cl}(\alpha_{S_1}) \cup \text{Cl}(\alpha_{S_2}) \subset \check{M}_{J,\infty}.$$

Hence, M_J is projectively diffeomorphic to the complete affine space. Thus, M_J is diffeomorphic to \mathbb{R}^3 , a contradiction to the assumption.

Now suppose that $A_1 = I_{S_1}^o - \{x\}$. Since

$$\text{Cl}(\alpha_{S_1}) \cup \text{Cl}(\alpha_{S_2}) \subset M_{J,\infty},$$

$S_1^o \cup A_1 \cup S_2^o$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and $M_J = S_1^o \cup A_1 \cup S_2^o$. Thus, M is finitely covered $\mathbb{S}^2 \times \mathbb{S}^1$ by Lemma 3.4. For $i = 1, 2$, S_i maps into a bihedron with a side $\mathbf{dev}_J(I_{S_i})$ in a great sphere \mathbb{S}^2 . S_1 and S_2 maps into the closures of two different components of $\mathbb{S}^3 - \mathbb{S}^2$. Again [the lifted developing map](#)

$$\mathbf{dev}_J|_{S_1^o \cup A_1 \cup S_2^o}$$

is an imbedding onto its image by geometry. Since $\mathbf{dev}_J(x)$ is an isolated boundary point, Corollary 2.7 implies the result. \square

(A)(ii) In this case, K^o is a nonempty properly convex open domain and $\text{bd}\Lambda(R) \cap M_J$ maps into $\text{bd}K$: Otherwise, we have $\dim K \leq n - 1$ and K is a subset of a hypersphere V . Then the two components of $H^o - V$ lift to cell imbeddings in $\Lambda(R)^o$ by equation (3.1). The closures of two cells in $\Lambda(R)$ are bihedral crescents again by equation (3.1). The two crescents are opposite. Thus, we are in case (i), a contradiction.

Here, we will show that there exists a toral π -submanifold.

Lemma 3.11. *Assume as in Theorem 3.8 with a concave affine 3-manifold N with compressible boundary. Suppose we are in case (A)(ii). Let K be the complement of $\mathbf{dev}_J(\Lambda_1(R))$. Then K is an unbounded subset of an affine space H^o . Moreover, $K \cap \partial H$ is a nonempty compact convex set, and $\text{bd}K \cap H^o$ is homeomorphic to a disk.*

Proof. Suppose that K is a bounded subset of H^o . Then $\mathbf{dev}_J|_{\Lambda_1(R)}$ is a homeomorphism to $H - K$ by equation (3.1). By Lemma 3.1, a component of $\text{bd}\Lambda_1(R) \cap M_J$ is not a sphere. There exists a noncompact component A_1 of $\text{bd}\Lambda(R) \cap M_J$ covering a closed surface B_1 . By Lemma 3.3, this is a contradiction as $h_J|_{\Gamma_N}$ is injective by equation (3.1). \square

We will show that $\text{bd}\Lambda_1(R) \cap M_J = \text{bd}\Lambda(R) \cap M_J$ precisely because of the nonexistence of the pair as in the condition of (i): Since for each crescent S , S^o is dense in S , we obtain

$$\text{bd}\Lambda(R) \cap M_J \subset \text{bd}\Lambda_1(R) \cap M_J.$$

Given a point $x \in \text{bd}\Lambda_1(R)$, choose a convex open neighborhood $B(x) \subset M_J$ with $\mathbf{dev}_J|_{B(x)}$ is an imbedding. $B(x) \cap S^o$ for a crescent S , $S \sim R$ is a closure of a component of $B(x) - I_S \cap B(x)$ for a totally disk $I_S \cap B(x)$ with boundary in $B(x)$. Thus, $B(x) \cap \Lambda_1(R)$ is a convex set. The set $B(x) - \Lambda_1(R)$ is a convex set K'' in $B(x)$. Since $\mathbf{dev}_J(\Lambda_1(R))$ is a homeomorphism to $H - K$ by equation 3.1,

- $\mathbf{dev}_J|_{B(x) \cap \Lambda_1(R)}$ maps homeomorphic to $\mathbf{dev}_J(B(x)) - K$;
hence,
- $\mathbf{dev}_J|_{B(x) - \Lambda_1(R)}$ maps homeomorphic to $\mathbf{dev}_J(B(x)) \cap K$.

Suppose that K'' has an empty interior. Then $\mathbf{dev}_J(K'')$ has an empty interior. Since $\mathbf{dev}_J(B(x)) \cap K \neq \emptyset$, there is a proper subspace P such that $K \subset P \cap H$ and $H^o - P$ is a union of two bihedrons disjoint from K . The inverse image of these in $\Lambda_1(R)$ are also bihedrons by equation 3.1. We take the closures. Then we have an [opposite](#) pair of bihedral 3-crescents with the interiors of the images disjoint from P . This is a contradiction.

Now K has a nonempty interior. The interior of K is disjoint from $\mathbf{dev}_J(T)$ for any crescent T , $T \sim R$ since otherwise

$$\mathbf{dev}_J(T^o) \cap K^o \neq \emptyset \text{ while } K^o \cap \mathbf{dev}_J(\Lambda_1(R)) = \emptyset.$$

Thus, $K^o \cap \Lambda(R) = \emptyset$, and $x \in \text{bd}\Lambda(R)$. Hence, we showed that

$$\text{bd}\Lambda_1(R) \cap M_J = \text{bd}\Lambda(R) \cap M_J.$$

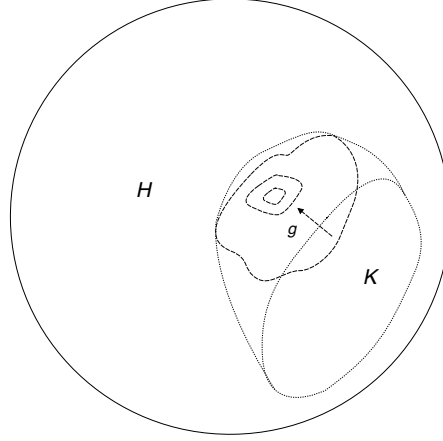


FIGURE 3. The diagram for H, K and $c_1 \subset \text{bd}K$ for the case (II)(A)(ii).

(a) The next step is to show that $\mathbf{dev}_J|_{\Lambda(R)} : \Lambda(R) \rightarrow H$ is injective:

If each component of $\text{bd}\Lambda(R) \cap M_J$ is simply connected, then ∂N is incompressible in N . Hence, a component A_1 of $\text{bd}\Lambda(R) \cap M_J$ is not simply connected and becomes compressible in N .

Now, $\text{bd}K \cap \text{bd}H$ is a convex domain in $\text{bd}K$ contractible to a point, and $\text{bd}K$ is homeomorphic to a sphere. Since

$$\text{bd}K \cap H^o = \text{bd}K - \text{bd}K \cap \text{bd}H$$

holds, $\text{bd}K \cap H^o$ is homeomorphic to a disk.

By compressibility, there exists a simple closed curve $c_1 \subset A_1$ so that $\mathbf{dev}_J(c_1)$ bounding a disk D_1 in $\text{bd}K \cap H^o$. Let Γ_1 be the subgroup of Γ_N acting on A_1 cocompactly. There exists an element $g \in \Gamma_1$ such that $g(c_1) \subset D_1 \cap A_1$ as in Section 3.2. We find an open disk D'_1 in $\Lambda_1(R)$ that is compactified by boundary c_1 . Then D'_1 bounds a 3-dimensional domain B_1 in $\Lambda_1(R) \cup A_1$ where B_1^o is a cell, and we may also assume that $g(B_1) \subset B_1^o$ disjoint from D'_1 by Lemma 2.4. Since D'_1 is separating in $\Lambda_1(R) \cup A_1$, $g^i(B_1)$ is a subset of B_1^o for every $i > 0$. Thus, we can find a fixed point x in K' for $g \in \Gamma_1$ by the Brouwer fixed point theorem. The premises of Proposition A.3 are satisfied for D_1 since K is convex. Proposition A.3 implies that $\mathbf{dev}_J(x)$ is the fixed point of the largest norm eigenvalue of $h_J(g)$ and the global attracting fixed point of $h_J(g)|H^o$.

Now, we prove the injectivity of $\mathbf{dev}_J|_{\Lambda(R)}$: Let $x_j, j = 1, 2$ be points of $\Lambda(R)$. Let $R_1, R_2 \sim R$ be two bihedral 3-crescents where $x_j \in R_j, j = 1, 2$. We may assume that $\mathbf{dev}_J(R_j)$ meets $\mathbf{dev}_J(\text{bd}\Lambda_1(R)) - \partial H$ by taking the maximal bihedral 3-crescents. Then $h_J(g)^i \mathbf{dev}_J(R_j)$ meets a neighborhood of $\mathbf{dev}_J(x)$ for sufficiently large i by equation (3.1). Since $\mathbf{dev}_J(g^i(I_{R_1}))$ and $\mathbf{dev}_J(I_{R_2})$ are very close containing nearby points for sufficiently large i and tangent to $\text{bd}K$, we obtain that $\mathbf{dev}_J(g^i(R_1))$ and $\mathbf{dev}_J(g^i(R_2))$ meet in the interior. By equation (3.1), we obtain

$$g^i(R_1)^o \cap g^i(R_2) \neq \emptyset$$

for sufficiently large i and hence

$$R_1^o \cap R_2^o \neq \emptyset.$$

By Theorem 5.4 and Proposition 3.9 of [19], $\mathbf{dev}_J|_{R_1 \cup R_2}$ is injective. Therefore, $\mathbf{dev}_J|_{\Lambda(R)}$ is injective. This completes the step (a).

(b) The next step is to show that $\text{bd}\Lambda(R) \cap M_J$ has a unique component: Since $\mathbf{dev}_J|_{\Lambda(R) \cap M_J}$ is injective, the restriction of an immersion $\mathbf{dev}_J|_{K' \cap \text{bd}\Lambda(R) \cap M_J}$ is a homeomorphism to its image Y in $\text{bd}K$. The set Y is an open surface. Then $Y/h_J(\Gamma_N)$ is a union of closed surfaces. Let Y_1 be the image of A_1 . $Y_1/h_J(\Gamma_1)$ is a connected closed surface homeomorphic to A_1/Γ_1 .

Since $\mathbf{dev}_J(x)$ is a unique attracting fixed point of $h_J(g)$ in H^o , $h_J(g)^i(c_1)$ goes into an arbitrary neighborhood of $\mathbf{dev}_J(x)$ in $\text{bd}K$ for sufficiently large i . $h_J(g)^i(c_1)$ goes into an arbitrary regular neighborhood of $\text{bd}K \cap \partial H$ in $\text{bd}K$ for sufficiently small negative number i . Using i and $-i$ for a large integer i , $h_J(g)^i(c_1)$ and $h_J(g)^{-i}(c_1)$ bound a compact annulus in $\text{bd}K \cap H^o$. If there is any other component \tilde{Y}_j of $Y \subset \text{bd}K$, then it lies in one of the annuli, a bounded subset of H^o ,

and \tilde{Y}_j covers a compact surface Y_j for some j . By Lemma 3.3, this is a contradiction. Thus, we obtain that

$$(3.2) \quad (\text{bd}K - \{x\}) \cap H^o = \mathbf{dev}_J(A_1).$$

Now, Γ_N acts on A_1 faithfully, properly discontinuously, and freely on an annulus A_1 and fixing the two ends of A_1 . This implies that Γ_N is virtually infinite-cyclic. The existence of g implies that the $h_J(\Gamma_N)$ fixes the unique point $\mathbf{dev}_J(x)$ corresponding to one of the end.

(c) We will need the following crescent R_P . Let $K_x \subset \mathbb{S}_x^2$ denote the subspace of directions of the segments with endpoints in $\mathbf{dev}_J(x)$ and K^o . Obviously, K_x is a convex open domain in an open half-space of \mathbb{S}_x^2 . Our $h_J(g)$ acts on K_x , and \mathbb{S}_x^2 has a $h_J(g)$ -invariant great circle \mathbb{S}^1 outside K_x as we can deduce by the existence of K_x .

We take a union of maximal segments in $\mathbf{dev}_J(\Lambda(R))$ from $\mathbf{dev}_J(x)$ in directions in \mathbb{S}^1 . Their union is a 2-hemisphere P with boundary in ∂H , and $\mathbf{dev}_J(x) \in P$.

By equation (3.1), we find an open bihedron $B \subset H - K$ whose boundary contains an open 2-hemisphere in ∂H and P . By taking an inverse and the closure, we obtain a bihedral 3-crescent $R_P \subset \Lambda(R)$ $x \in I_{R_P}$. By (a), g acts on R_P , I_P and x .

The last step is to show R_P has the desired property. By our choice of K_x and P , we obtain $\mathbf{dev}_J(I_{R_P})^o - \mathbf{dev}_J(x) \subset H - K^o$. By equation (3.2), we obtain

$$(3.3) \quad \mathbf{dev}_J(I_{R_P})^o - \mathbf{dev}_J(x) \subset \mathbf{dev}_J(A_1) \cup B.$$

Hence, $I_{R_P}^o - \{x\} \subset N_J$ for our bihedral 3-crescent R_P above. There is an element $g \in \Gamma_N$ acting on $R_P^o \cup I_{R_P}^o - \{x\}$. By Lemma 3.13, N contains a toral π -submanifold.

This finishes (A)(ii).

(B) Now suppose that $\Lambda(R)$ contains no triple of mutually overlapping bihedral 3-crescents S_i , $i = 1, 2, 3$, with $\mathbf{dev}_J(I_{S_i})$ in general position. By induction on overlapping pairs of bihedral 3-crescents, we obtain that $\mathbf{dev}_J(I_S)$ for a bihedral 3-crescent $S, S \sim R$ share a common point $q \in \partial H$ and hence its antipode $q_- \in \partial H$. Then $\Lambda(R)$ is a union of segments whose developing images end at q, q_- . The interior of such segments in $\Lambda(R)$ is called a *complete q -line*. Also, q -lines are subarcs of complete q -lines. $\text{bd}\Lambda(R) \cap M_J$ is foliated by subsets of q -lines.

Let \mathbb{S}_q^2 denote the sphere of directions of complete affine lines from q in H^o , and let \mathbb{S}_q^2 have a standard Riemannian metric of curvature 1. The space of q -lines in $\Lambda_1(R) \cap M_J$ whose developing image go from q to $-q$ is an open surface S_R with an affine structure. The developing

map \mathbf{dev}_J induces an immersion $\mathbf{dev}_{J,q} : S_R \rightarrow \mathbb{S}_q^2$. The surface S_R develops into a 2-hemisphere $H_q \subset \mathbb{S}_q^2$ whose interior H_q^o is identifiable with an affine 2-space. Denote by $\Pi_q : H^o \rightarrow H_q^o$ the projection.

The [Kuiper completion](#) \check{S}_R of S_R has an ideal subset c' that is the image of $\text{bd}\Lambda(R) \cap M_J$ and a geodesic boundary subset corresponding to $\delta_\infty\Lambda(R)$ and mapping to ∂H_q . We denote the extension by the same symbol $\mathbf{dev}_{J,q} : \check{S}_R \rightarrow \mathbb{S}_q^2$.

Let us consider $N_J = \Lambda(R) \cap M_J$ and we will use the restricted path-metric and complete it to obtain \check{N}_J . Each bihedral 3-crescent R in \check{M}_J with $R \cap M_J \subset N_J$ has a bihedral 3-crescent R' isometric to it in \check{N}_J under the distance nonincreasing map $\check{N}_J \rightarrow \check{M}_J$ by Proposition 2.14.

- Let us denote by $\Lambda(R') := \bigcup_{S \sim R'} S$ and
- define $\delta_\infty\Lambda(R') := \bigcup_{S \sim R'} \alpha_S$.

If each component of $\text{bd}\Lambda(R) \cap M_J$ is simply connected, then it is incompressible by Lemma 3.1. Thus, there is a component A_1 of $\text{bd}\Lambda(R) \cap M_J$ containing a simple closed curve c that is not null-homotopic in A_1 . We will use the same notation \mathbf{dev}_J for the extension of \mathbf{dev}_J to \check{N}_J . Let \mathcal{L}_q denote the set of complete q -lines l such that

$$l \subset R''' \text{ for } R''' \sim R, R''' \subset \check{N}_J, \text{ and } l \cap A_1 \neq \emptyset.$$

We define

$$A_{1+} := \bigcup_{l \in \mathcal{L}_q} l.$$

We claim that A_{1+} is homeomorphic to the injective image of a topologically open surface: Recalling the surface S_R above, we have a fibration $\Pi_R : \Lambda_1(R) \cap M_J \rightarrow S_R$ extending to $\check{N}_J \rightarrow \check{S}_R$, to be denoted by Π_R again. Π_R maps A_{1+} to a set c'' in the [ideal boundary](#) of \check{S}_R of S_R . Since q -complete lines pass the open surface A_1 foliated by q -arcs and A_1 is a smooth surface, $\mathbf{dev}_{J,q}|_{c''}$ maps locally injectively to an imbedded arc in H_q^o . Thus, c'' is locally an arc.

Suppose that two leaves l_1 and l_2 of A_{1+} go to the same point of an open arc α in c'' where $\mathbf{dev}_{J,q}|_\alpha$ is an imbedding. Since l_1 and l_2 are fibers, there is a point $((l))$ in S_R of \mathbf{d} -distance $< \epsilon$ from the images $((l_1)), ((l_2))$ of these lines in \check{S}_R . Inside $\Lambda_1(R)$, there exist paths of \mathbf{d} -length $< \epsilon$ from l_1 and l_2 to any point of a common line l in $\Lambda_1(R)$ corresponding to $((l))$ by spherical geometry. Taking $\epsilon \rightarrow 0$ and l closer to l_i , we obtain $l_1 = l_2$. Hence, we showed that A_{1+} fibers over c'' locally.

This implies that A_{1+} the image of an open surface locally. (However, globally it might be only injective image of an open surface.) We give a

new topology on A_{1+} by giving a basis of A_{1+} as the set of components of the inverse images of open sets in \check{M}_J . Then A_{1+} is homeomorphic to a smooth surface with this topology.

(i) As above, A_1 contains a simple closed curve c not homotopic to a point in A_1 . $c''' := \Pi_R(c)$ is either a compact arc, i.e., homeomorphic to an interval or a circle. Here, we will show that there exists a [toral \$\pi\$ -submanifold](#).

First, suppose that c''' is homeomorphic to an interval. Then c bounds an open disk D in the fibered space A_{1+} . Let Γ_1 be the subgroup of Γ_N acting on A_1 .

Then we can use a similar argument to (II)(A)(ii): First, there exists $g \in \Gamma_1$ so that $g(c)$ is in $D \cap A_1$ by Lemma 2.4. Hence g fixes a point x in D° that is a fixed point on A_{1+} by the Brouwer fixed point theorem.

Let x_q denote the complete q -line containing x in c'' . Let $g_q : \check{S}_R \rightarrow \check{S}_R$ be the induced map of $g : \Lambda(R) \rightarrow \Lambda(R)$. Recall the affine space H_q° . Consider x_q as the origin. Since the induced linear transformation $h(g)_q : H_q^\circ \rightarrow H_q^\circ$ is not trivial, its fixed point x_q in H_q° is locally isolated or there is a neighborhood of fixed points in the local arc $\mathbf{dev}_{J,q}(c'')$. The second case is not possible since $g(c)$ is in the disk bounded by c . Since g_q acts on the smooth arc c'' , g_q sends any sufficiently small open arc c'' containing x_q to a small open arc in c'' containing x_q . Since $h(g)_q$ is a non-trivial linear automorphism, $h(g)_q$ sends a small imbedded arc containing $\mathbf{dev}_{J,q}(x_q)$ to a small imbedded arc containing it. By the classification of the linear automorphism group of H_q° , we can show that $\mathbf{dev}_{J,q}(c'')$ is a properly imbedded arc in H_q° .

$$(3.4) \quad \begin{array}{ccc} A_{1+} & \xrightarrow{\Pi_R} & c'' \\ \downarrow \mathbf{dev}_J & & \mathbf{dev}_{J,q} \downarrow \\ H^\circ & \xrightarrow{\Pi_q} & H_q^\circ \end{array}$$

Since the left arrows of the above commutative diagrams are fibrations, $\mathbf{dev}_J|_{A_{1+}}$ properly imbeds to H° bounding a 3-dimensional domain \mathcal{D} where $\langle g \rangle$ acts on. In other words, $\mathbf{dev}_J(A_{1+})$ is a smoothly embedded proper surface in H° .

Since $h(g)_q$ is an affine transformation of H_q° fixing $\mathbf{dev}_{J,q}(x)$, $\mathbf{dev}_{J,q}(x)$ is a global attracting or repelling fixed point by the classification of linear maps acting on smooth properly imbedded arcs.

Let S_1 and S_2 be two bihedral crescents $\sim R$ meeting A_{1+} . Now, S_j , $j = 1, 2$, $S_j \sim R$, meeting A' can be moved by g^i to a bihedral 3-crescent $g^i(S_j)$ meeting D and passing a point arbitrarily close to x . Thus, $g^i(S_1)$ and $g^i(S_2)$ overlap as D is locally a convex disk containing

x in a convex ball mapping homeomorphic to a convex ball under \mathbf{dev}_J . Hence, $\mathbf{dev}_J|_{S_1 \cup S_2}$ is an imbedding to its image, and is injective by Theorem 5.4 and Proposition 3.9 of [19].

Let Λ' denote

$$\{S | S \sim R, S \cap A_{1+} \neq \emptyset\}.$$

Thus, $\mathbf{dev}_J|_{\Lambda'}$ is injective.

Since A_{1+} maps to the imbedded smooth arc $\mathbf{dev}_J(c'')$ invariant under g , the smooth surface $\mathbf{dev}_J(A_{1+})$ is closed and is tangent to ∂H at points of $\text{Cl}(A') \cap \partial H - \{q, q_-\}$. Let \mathcal{D} be the closure of the component $H^o - \mathbf{dev}_J(A_{1+})$ meeting the image of Λ' under \mathbf{dev}_J . For every open bihedron in \mathcal{D} in with a disk in the boundary contained in ∂H , the closure of the inverse image is in M_J is a bihedral crescent meeting $\mathbf{dev}_J(A_{1+})$. We obtain that $\mathbf{dev}_J|_{\Lambda'}$ is surjective to \mathcal{D} . Hence, $\mathbf{dev}_J|_{\Lambda'}$ is a homeomorphism to \mathcal{D} .

Choose a compact disk D_1 in $\Lambda' \cap M_J$ with boundary c . Then $D_1 \cup D$ bounds an open 3-ball B whose closure contains x . Then $g(D_1)$ is in B^o by connectedness of D_1 . We map these to their images in \mathcal{D} by \mathbf{dev}_J . Thus, $\mathbf{dev}_J(x)$ is the global attracting fixed point of $h_J(g)$ by Proposition A.3.

Now, $\mathbf{dev}_J(g^{-i}(c)) = h_J(g)^{-i}(\mathbf{dev}_J(c))$ must leave all compact subsets of H^o eventually and $g^i(c) \rightarrow \{x\}$ geometrically as $i \rightarrow \infty$. As above, we show using Lemma 3.3 that

$$(3.5) \quad A' := A_{1+} - \{x\} \subset M_J$$

by using c and its image $g^i(c)$ and $g^{-i}(c)$ in A_1 bounding an annulus.

Thus, we obtain

$$(3.6) \quad A' = \text{bd}\Lambda' \cap M_J \subset M_J.$$

Γ_N is again virtually infinite-cyclic since it acts on an annulus A' faithfully, properly discontinuously and freely and preserving two ends of A' . We choose a g -invariant crescent T with I_T meeting x as above in (A)(ii)(c). By Lemma 3.13, we obtain a toral π -submanifold from the bihedral 3-crescent T .

(ii) Suppose that $c'' = \Pi_R(c)$ is a circle. This case does not occur; we show that $\Lambda(R)$ is not maximal here.

Then the open surface A_{1+} is homeomorphic to an annulus A that is foliated by complete affine lines. Here, c is an essential simple closed curve. Let A_1 be a component of $A \cap M_J$ containing c . A_1 covers a compact submanifold in M . Let Γ_1 denote the group of deck transformations acting on A_1 . There exists an infinite-order element $g \in \Gamma_1$ sending c into a component of $A_1 - c$ by Lemma 2.4. Also, $g^i(c)$ goes

into any end neighborhood of A_1 for sufficiently large i since A_1/Γ_1 is a closed surface.

Recall the surface S_R that is the space of lines in $\Lambda(R) \cap M_J$. Recall that $\mathbf{dev}_{J,q}$ sends \check{S}_R to 2-hemisphere H_q where H_q^o is an affine subspace. $h_J(g)^i$ acts on a nontrivial closed curve $\mathbf{dev}_{J,q}(\Pi_R(c))$ bounded in an affine space H_q^o of \mathbb{S}_q^2 . Thus, $h_J(g)^i$ acts as an isometry on \mathbb{S}_q^2 with respect to a standard metric up to a choice of coordinates on \mathbb{S}_q^2 . Let $L(g)$ denote the linear part of $h_J(g)$ considered as an affine transformation of the affine space H^o . We obtain

$$h_J(g) = \begin{pmatrix} L(g) & v(g) \\ 0 & 1 \end{pmatrix}$$

where $v(g)$ is a 3-vector. By the classification of elements of $\mathbf{SL}_\pm(4, \mathbb{R})$, we have

- $L(g)$ has the direction vector v_q to q as an eigenvector,
- $L(g)$ induces an orthogonal linear map on $H_q^o := \mathbb{R}^3 / \langle v_q \rangle$. and
- we obtain g by post-composing with a translation in direction of q .

Thus, $v(g)$ is in the direction of v_q .

Suppose that $L(g)$ is parabolic with eigenvalues all equal to 1. By the second property above, $L(g)$ acts as identity on H_q^o , and g is a translation on each lines A_+ . Since $\langle g \rangle$ acts on A_+ , and as the identity on H_q^o , $h_J(g)$ is of form

$$(3.7) \quad \begin{pmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ for } \alpha, \beta \in \mathbb{R}, \gamma > 0.$$

If α, β are not all zero, then we can find a plane P_g of fixed points given by $\alpha y + \beta z + \gamma = 0$ in \mathbb{R}^3 . The inverse image of P_g in $\Lambda_1(R) \cap M_J$ is not empty. Since g acts properly on $\Lambda_1(R) \cap M_J$, this is absurd. Thus,

$$(3.8) \quad \alpha = 0, \beta = 0.$$

Otherwise, $L(g)$ is semisimple and $h_J(g)$ decomposes into

$$\begin{pmatrix} \lambda & 0 & \gamma \\ 0 & \mu O_g & 0 \\ 0 & 0 & \mu \end{pmatrix}, \lambda \mu^3 = 1, \lambda, \mu > 0$$

for an orthogonal 2×2 -matrix O_g . We deduce that either $g^i(c)$ either converges to a closed curve in the interior of A_{1+} or leaves all compact subsets as $i \rightarrow \infty$ or $i \rightarrow -\infty$.

Now, $A_{1+} \cap M_J$ has a component A_1 containing c .

Assume that $\mu \neq \lambda$. Then $g^i(c)$ geometrically converges to a compact closed curve in the interior of A_{1+} as $i \rightarrow \infty$ or $i \rightarrow -\infty$. Then the limit of $\mathbf{dev}_J(g^i(c))$ must be on a totally geodesic subspace P by the classification of elements of $\mathbf{SL}_{\pm}(4, \mathbb{R})$ passing $\mathbf{dev}(A_{1+})$. The inverse image of P in $\Lambda_1(R) \cap M_J$ diffeomorphic to an annulus. The group $\langle g \rangle$ should act properly discontinuously on the inverse image of P in $\Lambda_1(R) \cap M_J$. Since g^i is represented as uniformly bounded matrices on the projective space containing P for every $i \in \mathbb{Z}$, and g is of infinite order, this is a contradiction. Therefore, we have

$$(3.9) \quad \mu = \lambda = 1.$$

Since $h_J(g)(q) = q$, and h_J acts on H^o , it follows that $h_J(g)$ restricts to an affine transformation in H^o acting on the set of a parallel collection of lines. $h_J(g)$ acts as a translation composed with a rotation on H^o with respect to a Euclidean metric since the 3×3 -matrix of $L(g)$ decomposes into an orthogonal 2×2 -submatrix and the third diagonal element equal to 1.

Thus, in all cases as indicated by equations (3.8) or (3.9), g is of form

$$\begin{pmatrix} 1 & 0 & \gamma \\ 0 & O_g & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for an orthogonal 2×2 -matrix O_g .

Let L be an annulus bounded by c and $g(c)$ in A_{1+} . Since there is no bounded component of $L \cap M_J$ by Lemma 3.3, L is a subset of M_J . There exists an open neighborhood N of L in M_J , and $\bigcup_{i \in \mathbb{Z}} g^i(N) \subset M_J$ covers A_{1+} . Thus, we conclude $A_{1+} \subset M_J$.

By restricting a Euclidean metric H^o , we obtain a Euclidean metric on an open set in M_J containing $\Lambda(R) \cap M_J$ and $\bigcup_{i \in \mathbb{Z}} g^i(L)$. We obtain a closed set

$$\Lambda' \subset \bigcup_{i \in \mathbb{Z}} g^i(N) \cup \Lambda(R) \cap M_J,$$

that is foliated by complete q -lines and

$$\Lambda(R) \cap M_J \subset \Lambda'$$

properly. Also, Λ' contains a ϵ -neighborhood of $\Lambda(R) \cap M_J$ in the Euclidean metric.

The subspace Λ' fibers over the surface Σ of complete q -lines in Λ' as before in the beginning of (B). Then the [Kuijer completion](#) $\check{\Sigma}$ has an affine structure. We extend the above fibration $\Pi_R : \Lambda_1(R) \cap M_J \rightarrow S_R$ to $\Lambda' \rightarrow \Sigma$ to be denoted by Π_R again. Σ properly contains S_R bounded by the arc α corresponding to A_{1+} . We take a short geodesic k in Σ

connecting the end points of the short subarc α_1 in α so that they bound a disk in Σ . k can be extended until it ends in the ideal set of $\tilde{\Sigma}$ corresponding to complete q -lines in $\delta_\infty \Lambda(R)$. We choose a 2-dimensional crescent S'' in $\tilde{\Sigma}$ bounded by k containing the image of S° in Σ and containing α_1 . (See the maximum property in Section 6.2 of [20].) The union of complete q -lines through S'' is in Λ' if we take k sufficiently close to $\Lambda_1(R) \cap M_J$. The inverse image \hat{S} of S'' in M_J is a bihedral 3-crescent in Λ' properly containing S . This contradicts the maximality of $\Lambda(R)$, which is a contradiction to how we defined $\Lambda(R)$ in equation (2.1). \square

3.4. Toral π -submanifolds.

Lemma 3.12. *A toral π -submanifold N of type I is homeomorphic to a solid torus or solid Klein-bottle and is a concave affine 3-manifold of type I.*

Proof. First by Lemma 3.1, there is no boundary component of N homeomorphic to a sphere or a real projective plane.

By Definition 3.6, N is covered by $R^\circ \cup I_R^\circ - \{x\}$ for a hemispherical crescent R and $x \in I_R$ and hence is a concave affine 3-manifold of type I.

Since the deck transformation group acts on the annulus $I_R^\circ - \{x\}$ properly discontinuously and freely, the group is isomorphic to a virtually infinite-cyclic group. By Theorem 5.2 of [30], a toral π -submanifold is homeomorphic to a solid torus or a solid Klein bottle. \square

Lemma 3.13. *Suppose that \check{M}_J has no hemispherical 3-crescent. (See Hypothesis 2.10.) Let N be a concave affine 3-manifold of type II in M covered by $\Lambda(R) \cap N_J$ for a bihedral crescent R . We suppose that*

- *The Kuiper completion \check{N}_J of some cover N_J of the holonomy cover of N contains a toral bihedral 3-crescent S fixing a point $x \in I_S^\circ$ as an attracting fixed point.*
- *The deck transformation group of N is virtually infinite-cyclic.*
- *$\text{dev}_J : \Lambda(R) \cap N_J \rightarrow H - K^\circ$ is an imbedding to its image containing $H - K$ for a compact convex domain K in the 3-hemisphere H with $K^\circ \subset H^\circ$, $K^\circ \neq \emptyset$.*

Then N contains a unique toral π -submanifold of type II, homeomorphic to solid torus or solid Klein-bottle, and the interior of every bihedral 3-crescent in \check{N}_J meets the inverse image of the toral π -submanifold in N_J .

Proof. First by Lemma 3.1, there is no boundary component of N homeomorphic to a sphere or a real projective plane.

By definition, $N_J \subset \Lambda(R)$ for a bihedral 3-crescent R . Let Γ_J denote the group of deck transformation acting on N_J so that $N = N_J/\Gamma_J$. We have a toral bihedral 3-crescent R in \check{N}_J .

By assumption, Γ_J is virtually infinite-cyclic. Two bihedral 3-crescents R_1 and R_2 are not [opposite](#) since $K^o \neq \emptyset$ holds. Let R_1 and R_2 be two [toral bihedral 3-crescents](#) such that $R_1, R_2 \sim R$. Let R'_i denote $R_i^o \cup I_{R_i}^o - \{x_i\}$ for a fixed point x_i of the action of an infinite order generating deck transformation g_i acting on R_i so that $R'_i/\langle g_i \rangle$ is homeomorphic to a solid torus. Let F_i , $i = 1, 2$, denote the compact fundamental domain of R'_i . Then the set

$$G_i := \{g \in \Gamma_J | g(F_i) \cap F_i \neq \emptyset\}, i = 1, 2,$$

is finite. We can take a finite index normal subgroup Γ' of the virtually infinite-cyclic group Γ_J so that $\Gamma' \cap G_i := \{e\}$ for both i . Then a cover of the compact submanifold $R'_i/\langle g_i \rangle$ imbeds in N_J/Γ' . Thus, there is some cover N_1 of N so that these lift to embedded submanifolds.

We denote these in N_1 by T_1 and T_2 . We may assume that $T_i = R'_i/\langle g_i \rangle$. x_i is the *fixed point* of R_i and g_i acts on R'_i .

Suppose that they overlap. Then $R_1 \cap R_2$ is a component of $R_1 - I_{R_2}$ by Theorem 5.4 of [19]. Considering $T_1 \cap T_2$ that must be a solid torus not homotopic to a point in each T_i , we obtain that a nonzero power of g_1 and a nonzero power of g_2 are equal. Therefore, $x_1 = x_2$ and g_i fixes the point $x_1 = x_2$.

We say that two toral bihedral 3-crescents R_1 and R_2 are *equivalent* if they overlap in a cover of a solid torus in N_J . This generates an equivalence class of solid π -tori. We write $R_1 \cong R_2$.

By this condition, $x = x_i$ for every fixed point x_i of a toral bihedral 3-crescent R_i , $R_i \cong R$. x is fixed by g_i for all i . Let S be a toral bihedral 3-crescent in \check{N}_J . We define

$$\hat{\Lambda}(S) := \bigcup_{R \cong S} R, \quad \delta_\infty \hat{\Lambda}(S) := \bigcup_{R \cong S} \alpha_R.$$

Now $\hat{\Lambda}(S') \cap N_J$ covers a compact submanifold in N : Let T be any bihedral 3-crescent in \check{N}_J where g acts with x as an attracting fixed point. Then $T - \text{Cl}(\alpha_T) - \{x\} \subset N_J$ as in cases (A)(ii) or (B)(i).

Since we have no [two-faced submanifolds](#), we see that either

$$\hat{\Lambda}(S) = g(\hat{\Lambda}(S)) \text{ or } \hat{\Lambda}(S) \cap g(\hat{\Lambda}(S)) \cap N_J = \emptyset \text{ for } g \in \Gamma_J.$$

(This follows as in Lemma 7.2 of [19].) We can also show that the collection

$$\{g(\hat{\Lambda}(S) \cap N_J) | g \in \Gamma_J\}$$

is locally finite in M_J as we did for $\Lambda(S) \cap M_h$ in Chapter 9 of [19].

Hence, the image of $\hat{\Lambda}(S) \cap N_J$ is closed in M , and it is a compact submanifold.

Since $\hat{\Lambda}(R)$ is a union of segments from x to

$$\delta_\infty \hat{\Lambda}(R) := \delta_\infty \Lambda(S) \cap \hat{\Lambda}(R),$$

we have $\text{bd} \hat{\Lambda}(R) \cap N_J$ is on a union L of such segments from x to $\text{Cl}(\delta_\infty \hat{\Lambda}(R))$ passing the set. The open line segments are all in N_h as they are in toral bihedral 3-crescents. Since $\hat{\Lambda}(R)$ is canonically defined, the virtually infinite-cyclic group Γ_J acts on the set. Also, $\hat{\Lambda}(R) \cap N_J$ is connected since we can apply the above paragraph to 3-crescents in $\hat{\Lambda}(R)$ also.

The interior of $\hat{\Lambda}(R) \cap M_J$ is a union of open segments from x to an open surface $\delta_\infty \hat{\Lambda}(R)$. The surface cannot be a sphere or a real projective plane since a toral π -submanifold has boundary. Since $\delta_\infty \hat{\Lambda}(R)$ is the complement of ∂H of a compact convex set, it is thus homomorphic to a 2-cell. Therefore, the interior of $\hat{\Lambda}(R) \cap M_J$ is homeomorphic to a 3-cell.

We showed that $\hat{\Lambda}(R) \cap M_J$ covers a compact submanifold in N in the proof of Lemma 3.13.

Since N has the virtually infinite-cyclic fundamental holonomy group, we obtain that the holonomy group image of the deck transformation group acting on $\hat{\Lambda}(R) \cap M_J$ is virtually infinite-cyclic. Since the holonomy homomorphism is injective, the deck transformation group acting on $\hat{\Lambda}(R) \cap M_J$ is virtually infinite-cyclic.

Since a toral π -submanifold is covered by a cell, and has the fundamental group that is a virtually infinite-cyclic group, it is covered by a solid torus by Theorem 4.15 of [30]. As in the proof of Lemma 3.12, a toral π -submanifold is homeomorphic to a solid torus or a solid Klein bottle.

Now, we go to the final part: We assumed that $\mathbf{dev}_J|_{\Lambda(S) \cap N_J}$ is an injective map into the complement of a convex domain in H . Thus $\mathbf{dev}_J(\hat{\Lambda}(R) \cap N_J)$ is a complement of a domain K' in H where $K' \subset K$ and the closure of K' is convex. Given any bihedral 3-crescent R_1 in $\Lambda(S)$, suppose that the open 3-bihedron $\mathbf{dev}_J(R_1^\circ)$ does not meet $\mathbf{dev}_J(\hat{\Lambda}(R))$. Then $\mathbf{dev}_J(\alpha_{R_1})$ and $\mathbf{dev}_J(\alpha_T)$ for a toral bihedral 3-crescent T , $T \cong R$, have to be 2-hemispheres in ∂H antipodal to each other. Let g_T denote the deck transformation acting on $T^\circ \cup I_T^\circ - \{x\}$ for an attracting fixed point x of g_T . Then

$$g_T^i(R_1) \subset g_T^j(R_1) \text{ for } i < j$$

by Proposition 3.9 of [19] since their images overlap and the image of the latter set contains the former one and $\mathbf{dev}_J|_{N_J}$ is injective. Hence, $\bigcup_{i \in \mathbb{N}} g_T^i(R_1)$ contains R_1 and its closure R is another toral bihedral 3-crescent since g_T acts on it and by equations (3.5) and (3.6). Then T and R are **opposite**. This is a contradiction since then K has to have an empty interior. We assumed otherwise in the premise. \square

Remark 3.14. *By Lemma 3.13, if toral π -submanifolds have totally geodesic or empty boundary, then they are classified by Theorem 11.2 in [17] as affine suspensions of a 2-hemisphere, a real projective plane, a real projective sphere, or a π -annulus (or π -Möbius band) of type C. The “concavity” of the boundary gives us some difficulty in classifying these.*

3.5. Sharpening of Theorems 3.7 and 3.8. A **concave affine 3-manifolds** are always prime.

Theorem 3.15. *Let N be a concave affine 3-manifold in a compact real projective 3-manifold M . Suppose that M is not complete affine or bihedral. Assume that M has no two-faced submanifold. Then N is irreducible or M is an affine Hopf 3-manifold.*

Proof. Let N_J denote a component of the inverse image of N in M_J .

Suppose that N is a concave affine 3-manifold of type I. If ∂N is incompressible in N , then N_J is a union of an open 3-hemisphere R° with a disk I_R° in ∂H where R is a hemispherical crescent. Hence, N_J is irreducible.

Suppose that ∂N is compressible. Then we showed in the proof of Theorem 3.7 that N_J equals $R^\circ \cup I_R^\circ - \{x\}$ for a point $x \in I_R^\circ$ for a 3-hemispherical crescent R and I_R a 2-hemisphere in ∂R as in the proof of part (I) of Theorem 3.3. Clearly, this is an irreducible 3-manifold.

Suppose that N is a concave affine 3-manifold of type II. We assume that M_J has no hemispherical crescent. (See Hypothesis 2.10.) We follow the proof of Theorem 3.7. We divide into cases (A) and (B).

Let R be a bihedral 3-crescent so that $N_J \subset \Lambda(R)$.

- (A) Suppose that we have three mutually overlapping bihedral 3-crescents R_1, R_2 , and R_3 with $\{I_{R_i} | i = 1, 2, 3\}$ in general position.

We can have two possibilities:

- (i) Suppose that there exists a pair of **opposite** bihedral 3-crescents $S_1, S_2 \sim R$.
- (ii) There is no such pair of bihedral 3-crescents.

By Lemma 3.10, (i) implies that M is an affine Hopf manifold. We now work with (ii) only. We now work with (ii) only. By Lemma 3.1, $\text{bd}\Lambda(R) \cap M_J$ has no sphere boundary.

Any 2-sphere in $\Lambda(R) \cap M_J$ can be isotoped into $\Lambda_1(R) \cap M_J$. Recall that we showed using Lemma 3.3 in the beginning of (A)(ii) in the proof of Theorem 3.8 that K cannot be bounded in H° . We have $K \cap \partial H \neq \emptyset$. Since $K \cap \partial H$ is a contractible compact set,

$$\text{bd}K \cap H^\circ = \text{bd}K - K \cap \partial H$$

is an open disk separating $H^\circ - K$ with K° . By the Van Kampen theorem, $H^\circ - K$ has trivial homotopy groups in dimensions one and two. Thus, $\Lambda_1(R) \cap M_J$ is contractible and every immersed sphere is null-homotopic.

Now we go to the case (B) in the proof of Theorem 3.8 where $\Lambda(R)$ is a union of the segments whose developing image end at the antipodal pair q, q_- . Since $\Lambda(R) \cap N_J$ fibers over an open surface S with fiber homeomorphic to real lines, N is irreducible. \square

3.6. Proof of Theorem 1.3.

Proof. Given a compact real projective 3-manifold M with empty or convex boundary, if M has a compressible component of a two-faced totally geodesic submanifold of type I, then M is an affine Hopf manifold by Theorem 3.2.

Now suppose that M is not an affine Hopf manifold. We split along the union of all two-faced totally geodesic submanifolds of type I now to obtain M^s . Theorem 3.7 imply the result.

To complete, we repeat the above argument for two-faced totally geodesic submanifolds of type II and Theorem 3.8. \square

4. TORAL π -SUBMANIFOLDS AND THE DECOMPOSITION

We will now refine the above results.

Theorem 4.1. *Suppose that a compact real projective 3-manifold M with empty or convex boundary. Suppose that M is not complete affine or bihedral and M is not an affine Hopf 3-manifold. Suppose that M has no two-faced submanifold of type I, and M has no concave affine 3-manifold of type I with boundary incompressible to itself. Suppose that M' below has no two-faced submanifold of type II, and M' has no concave affine 3-manifold of type II with boundary incompressible to itself. Then the following hold:*

- each concave affine submanifold of type I in M with compressible boundary contains a unique toral π -submanifold T of type I where T has a compressible boundary.

– There are finitely many disjoint toral π -submanifolds

$$T_1, \dots, T_n$$

obtained by taking one from concave affine submanifolds in M with compressible boundary.

- We remove $\bigcup_{i=1}^n \text{int} T_i$ from M . Call M' the resulting real projective manifold with convex boundary.

– Each concave affine submanifold of type II in M' with compressible boundary contains a unique toral π -submanifold T of type II where T has a compressible boundary.

– There are finitely many disjoint toral π -submanifolds

$$T_{n+1}, \dots, T_{m+n}$$

obtained by taking one from concave affine submanifolds in M' with compressible boundary.

- $M - \bigcup_{i=1}^{n+m} \text{int} T_i$ is 2-convex.

Proof. If N is a concave affine 3-manifold of type I with a compressible boundary into N , then its universal cover is in a hemispherical 3-crescent and N is homeomorphic to a solid torus and is a toral π -submanifold by Lemma 3.12. These concave affine 3-manifolds are mutually disjoint.

We remove these and denote the result by M' . Then $M - \bigcup_{i=1}^n \text{int} T_i$ has totally geodesic boundary. The cover M'_J of M' is given by removing the inverse images of T_1, \dots, T_n from M_J . We complete M'_J to \tilde{M}'_J . Now we consider when N is a concave affine 3-manifold arising from bihedral 3-crescents in \tilde{M}'_J . We obtain toral π -submanifold II in N by Lemma 3.13.

From M' we remove the union of the interiors of toral π -submanifolds T_n, \dots, T_{n+m} . Then $M - \bigcup_{i=1}^{n+m} \text{int} T_i$ has a convex boundary as P_i has concave boundary.

We claim that this manifold $M - \bigcup_{i=1}^{n+m} \text{int} T_i$ is 2-convex. Suppose not. Then by Theorem 1.1 of [19], we obtain again a 3-crescent R' in the Kuiper completion of $M_J - p_J^{-1}(\bigcup_{i=1}^{n+m} \text{int} T_i)$. The set maps to \tilde{M}_J , and the image R'' of R' is a 3-crescent by Proposition 2.14. The 3-crescent R'' has the interior disjoint from ones we already considered. However, Theorem 3.8 shows that R''^o must meet the inverse image $p_J^{-1}(\bigcup_{i=1}^n \text{int} T_i)$, which is a contradiction.

Lemma 3.13 shows that each T_i is homeomorphic to a solid torus or a solid Klein bottle.

□

Proof of Theorem 1.4. We may assume that M is not complete or bi-hedral since then M is convex and the conclusions are true. As stated, \check{M}_J does not contain any hemispherical 3-crescent.

Now M^s decomposes into concave affine manifolds of type II with boundary compressible or incompressible to themselves. Theorem 0.1 of [18] shows that a 2-convex affine 3-manifold is covered by a cell and hence is irreducible. Toral π -submanifolds and concave affine 3-manifolds of type II with incompressible boundary are irreducible by Lemma 3.13 and Theorem 3.15. Hence, the decomposed submanifolds are prime.

□

APPENDIX A. CONTRACTION MAPS

Here, we will discuss contraction maps in \mathbb{R}^n . A *contracting map* $f : X \rightarrow X$ for a metric space X with metric d is a map so that $d(f(x), f(y)) < d(x, y)$ for $x, y \in X$.

Lemma A.1. *A linear map L has the property that all the norms of the eigenvalues are < 1 . if and only if L is a contracting map for the distance induced by a norm.*

Proof. See Corollary 1.2.3 of Katok [35].

□

Proposition A.2. *$\langle g \rangle$ acts on $\mathbb{R}^n - \{O\}$ (reps. $H - \{O\}$ for a half space $H \subset \mathbb{R}^n$) properly if and only if the all the norms of the eigenvalues of g are > 1 or < 1 .*

Proof. Suppose that $\langle g \rangle$ acts on $\mathbb{R}^n - \{O\}$ properly. For a sphere $S = \mathbb{S}^{n-1}$, we have $g^n(S)$ is inside a unit ball B for some integer n by properness of the action. This implies that $g^n(B) \subset B$, and the norms of the eigenvalues of g^n are < 1 . The case of the half space H is similar.

For the converse, by replacing g with g^{-1} if necessary, we assume that all norms of eigenvalues < 1 . Lemma A.1 implies the result. □

Proposition A.3. *Let D be a domain in $\mathbb{R}^n \subset \mathbb{S}^n$. Let g be a projective automorphism of \mathbb{S}^n acting on D and \mathbb{R}^n . We assume the following:*

- S is a compact connected subset of D so that $D - S$ has two components D_1 and D_2 where D_1 is bounded,
- g acts on D with a fixed point $x \in \mathbb{R}^n$ in the closure of D_1 .
- $g(S) \subset D_1$.
- Every complete affine line containing x meets S at at least one point.

- $D_1 \subset \{x\} * S$ where $\{x\} * S$ is the union of all segments from x ending at S .

Then x corresponds to the 1-dimensional subspace of the largest eigenvalue of g and is the global attracting fixed point of g in \mathbb{R}^n .

Proof. Choose the coordinate system so that x is the origin. Let $L(g)$ denote the linear part of the g in this coordinate system. Suppose that there is a norm of the eigenvalues of $L(g)$ equal to 1. Then there is a real eigensubspace V of dimension 1 or 2 associated to an eigenvalue of norm ≥ 1 . We have $S_V := V \cap S \neq \emptyset$ by the fourth assumption. The set $\Theta(S_V)$ of directions of S_V from x is $L(g)$ -invariant and is either the set of a point, the set of a pair of antipodal points, or a circle. Since V has an invariant metric, there is a point t of S_V where a maximal radius of S_V takes place. Then $g(t) \in g(S_V)$ must meet $D_2 \cup S_V$, a contradiction.

Thus, the norms of eigenvalues of $L(g)$ are < 1 . By Lemma A.1, $L(g)$ has a fixed point x as an attracting fixed point. The conclusion follows. \square

APPENDIX B. THE BOUNDARY OF A CONCAVE AFFINE MANIFOLDS ARE NOT STRICTLY CONCAVE.

The following generalizes the maximum property in Section 6.2 of [20].

Theorem B.1. *Let N be a concave affine 3-manifold of type II in M . Then for every $y \in \partial N$, y has no tangent totally geodesic open disk D containing y and $D^\circ \subset N^\circ$.*

Proof. Let M_J be a cover as in the main part of the paper. Suppose that we have a disk D as above. Then if y is a boundary point of M_J , then D must be in ∂M_J by geometry. This contradicts the premise.

Suppose that N is covered by $\Lambda(R) \cap M_J$. Since y is not a boundary point of M_J , we take a convex compact neighborhood $B(y)$ of the convex point y so that $\mathbf{dev}_J(B(y))$ is an ϵ - \mathbf{d} -ball for some $\epsilon > 0$. Then $B(y) - \Lambda(R)$ is a properly convex domain with the image $\mathbf{dev}_J(B(y) - \Lambda(R))$ is properly convex. For each point $z \in \text{bd}\Lambda(R) \cap B(y)$, let S_z , $S_z \sim R$, be a bihedral crescent containing z . Since $\Lambda(R)$ is maximal, $\mathbf{dev}_J(I_{S_z})$ is a supporting plane at $\mathbf{dev}_J(z)$ of $\mathbf{dev}_J(B(y) - \Lambda(R))$.

We perturb a small convex disk $D \subset I_{S_y}$ containing y away from y , so that the perturbed convex disk D' is such that the closure of $D' \cap B(y) - \Lambda(R)$ is a small compact disk D'' with

$$\partial D'' \subset \text{bd}\Lambda(R) \cap M_J \text{ and } D''^\circ \cap \Lambda(R) = \emptyset.$$

Then $\partial D''$ bounds a compact disk B' in $\text{bd}\Lambda(R) \cap B(y)$. Choose a point z_0 in the interior of D'' . For each point $z \in B'$, we choose a maximal segment $s_z \subset S_z$ starting from z_0 passing z ending at a point $\delta_+ s_z$ of α_{S_z} . We obtain a compact 3-ball $B_{z_0} = \bigcup_{z \in B'} s_z$ with boundary in $\delta_\infty \Lambda(R)$. The boundary is the union of $D_{z_0} := \bigcup_{z \in \partial D''} s_z$, a compact disk, and an open disk

$$\alpha_{z_0} := \bigcup_{z \in B_{z_0}^\circ} \delta_+ s_z \subset \delta_\infty \Lambda(R).$$

By taking D' sufficiently close to D , we obtain that $\mathbf{dev}_J(D_{z_0})$ and $\mathbf{dev}_J(\alpha_{z_0})$ are sufficiently close to $\mathbf{dev}_J(I_{S_y})$ and $\mathbf{dev}_J(\alpha_{S_y})$ respectively. Hence, $\mathbf{dev}_J(B_{z_0})$ is a bihedron and B_{z_0} is a bihedral crescent.

Since B_{z_0} is a crescent $\sim S_y$, $S_y \sim R$, we have $B_{z_0} \subset \Lambda(R)$. This contradicts our choice of y and D'' . \square

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